Coulomb Branches in Geometry and Representation Theory

(From Seiberg-Witten to BZSV Sec. 8)

Harold Williams, USC

Workshop on Relative Langlands Program

National University of Singapore, 1/7/26 - 1/9/26

Overview

- Coulomb branches are geometric objects associated to supersymmetric quantum field theories.
- First named by Seiberg-Witten in '94, they have since been connected to a wide range of mathematics.
- These lectures will survey how the subject developed from '94 to today, culminating with the appearance of Coulomb branches in the work of Ben-Zvi-Sakellaridis-Venkatesh.
- Warning: we will mention some key highlights from this history, but the resulting theory is ultimately the work of more physicists and mathematicians than we could hope to name!

Crash Course in QFT

- Classically, a d-dimensional field theory is given by
  1) a manifold M (spacetime), generally w/ a metric
  2) a space of fields $\mathcal{F}$, local objects over M
  3) a local action functional $S : \mathcal{F} \to \mathbb{R}$

Ex G compact group $\rightsquigarrow$ Yang-Mills theory:

$\mathcal{F} = \{$ principal G-bundle $P \to M$ w/ connection $A \}$

$S : A \mapsto \int_M tr(F_A \wedge * F_A)\, dM \quad (F_A \in \mathfrak{g} \otimes \Omega^2(M)$ curvature$)$

Crash Course in QFT

- Classically, a d-dimensional field theory is given by
    1) a manifold M (spacetime), generally w/ a metric
    2) a space of fields $\mathcal{F}$, local objects over M
    3) a local action functional $S : \mathcal{F} \to \mathbb{R}$

<u>Ex</u> G compact group $\rightsquigarrow$ Yang-Mills theory:

$\mathcal{F} = \{$ principal G-bundle $P \to M$ w/ connection $A \}$

$S : A \mapsto \int_M tr(F_A \wedge * F_A)\, dM \qquad (F_A \in \mathfrak{g} \otimes \Omega^2(M)$ curvature$)$

- Measurements in the corresponding quantum theory are heuristically given by path integrals

<u>Ex</u> Given a local observable $\mathcal{O}_x : \mathcal{F} \to \mathbb{R}$ at $x \in M$, its expectation value is

measure not well-defined

$$\langle \mathcal{O}_x \rangle = \int_{\phi \in \mathcal{F}} \mathcal{O}_x(\phi)\, e^{-iS(\phi)/\hbar} \text{``}d\phi\text{''}$$

Crash Course in QFT

- Fundamental case: $M = \mathbb{R}^{d-1,1}$ (or $M = \mathbb{R}^d$)

- In a relativistic theory the action of the Poincaré algebra $\mathfrak{p}_0 = \mathfrak{so}(d-1,1) \ltimes \mathbb{R}^{d-1,1}$ on M is lifted to a compatible action on $\mathcal{F}$.

- In a supersymmetric theory this is further lifted to an action of a super Lie algebra $\mathfrak{p}_0 \oplus \mathfrak{p}_1$.

Crash Course in QFT

- Fundamental case: $M = \mathbb{R}^{d-1,1}$ (or $M = \mathbb{R}^d$)

- In a relativistic theory the action of the Poincaré algebra $\mathfrak{p}_0 = \mathfrak{so}(d-1,1) \ltimes \mathbb{R}^{d-1,1}$ on M is lifted to a compatible action on $\mathcal{F}$.

- In a supersymmetric theory this is further lifted to an action of a super Lie algebra $\mathfrak{p}_0 \oplus \mathfrak{p}_1$.

- Possible $\mathfrak{p}_1$ classified by $\mathcal{N} = \#\, \mathfrak{so}(d-1,1)$-irreps in $\mathfrak{p}_1$

- We will be interested in the cases $d=4$, $\mathcal{N}=2$ and $d=3$, $\mathcal{N}=4$, both characterized by $\dim_{\mathbb{R}} \mathfrak{p}_1 = 8$.

- Such theories are more complicated to write down than ones with less supersymmetry, but more tractable to analyze.

Crash Course in QFT

- Simplest examples: compact group G and complex G-rep V ⇝ 4d $\mathcal{N}=2$ gauge theory $\mathcal{Q}^{4d}_{G,V}$ and 3d $\mathcal{N}=4$ gauge theory $\mathcal{Q}^{3d}_{G,V}$
- Fields include a G-connection and a section of a V-bundle, plus a package of other fields.
- $\mathcal{Q}^{4d}_{G,0}$ is also called super-Yang-Mills theory, and $\mathcal{Q}^{4d}_{SU(n),(\mathbb{C}^n)^k}$ is super-QCD w/ n colors and k flavors.

Crash Course in QFT

- Simplest examples: compact group G and complex G-rep V $\rightsquigarrow$ 4d $\mathcal{N}=2$ gauge theory $\mathcal{Q}^{4d}_{G,V}$ and 3d $\mathcal{N}=4$ gauge theory $\mathcal{Q}^{3d}_{G,V}$
- Fields include a G-connection and a section of a V-bundle, plus a package of other fields.
- $\mathcal{Q}^{4d}_{G,0}$ is also called super-Yang-Mills theory, and $\mathcal{Q}^{4d}_{SU(n),(\mathbb{C}^n)^k}$ is super-QCD w/ n colors and k flavors.
- For noncompact M, we should specify the asymptotics of fields appearing in a path integral.
- Possible choices correspond classically to vacua, energy-minimizing field configurations
- Many theories have a unique vacuum, but susy theories generally have a moduli space of vacua.

The Seiberg-Witten Solution

- In two '94 papers, Seiberg-Witten "solved" 4d $\mathcal{N}=2$ SU(2) SYM and SQCD by analyzing global properties of their moduli of vacua.

- A landmark achievement for many reasons:

The Seiberg-Witten Solution

- In two '94 papers, Seiberg-Witten "solved" 4d $\mathcal{N}=2$ $SU(2)$ SYM and SQCD by analyzing global properties of their moduli of vacua.

- A landmark achievement for many reasons:

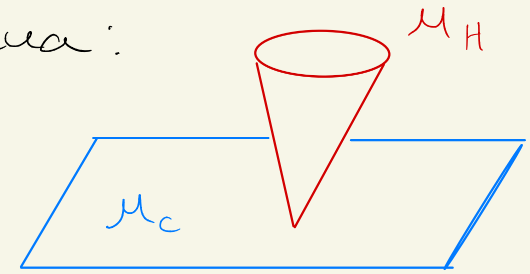
  1) (Physics) First physical derivation of quark confinement via monopole condensation.

  2) (Topology) Led to replacement of Donaldson 4-mfld invariants by more tractable SW invariants.

  3) (Geometry) Their solution introduced an explicit connection btwn susy QFT and algebraic geometry that subsequent developments built on.

The Seiberg-Witten Solution

- For any 4d $\mathcal{N}=2$ or 3d $\mathcal{N}=4$ gauge theory $\mathcal{Q}$ on a flat spacetime, they distinguished two extremal components of the moduli of vacua:



  1) the Coulomb branch $\mathcal{M}_C$
  2) the Higgs branch $\mathcal{M}_H$

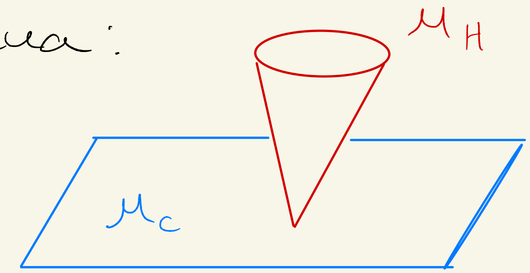
- Easy facts: 1) $\mathcal{M}_C(\mathcal{Q}^{4d}_{G,V}, \mathbb{R}^4) \simeq \mathbb{C}^{\mathrm{rk}\, G}$
  2) $\mathcal{M}_H$ only depends on G and V (i.e. not M)
  3) $\mathcal{M}_H = pt$ if $V = 0$

The Seiberg-Witten Solution

- For any 4d $\mathcal{N}=2$ or 3d $\mathcal{N}=4$ gauge theory $\mathcal{Q}$ on a flat spacetime, they distinguished two extremal components of the moduli of vacua:

  1) the Coulomb branch $\mathcal{M}_C$

  2) the Higgs branch $\mathcal{M}_H$



- Easy Facts: 1) $\mathcal{M}_C(\mathcal{Q}^{4d}_{G,V}, \mathbb{R}^4) \simeq \mathbb{C}^{\mathrm{rk}\, G}$

  2) $\mathcal{M}_H$ only depends on G and V (i.e. not M)

  3) $\mathcal{M}_H = pt$ if $V=0$

- It was known that for generic $v \in \mathcal{M}_C$, the low-energy behavior of $\mathcal{Q}$ (i.e. its behavior after integrating out high energy fields) was approximated by an abelian gauge thy determined by:

  1) "electric" coordinates $a_1, \ldots, a_r$ in a nbhd U_v,

  2) a prepotential $\mathcal{F}: U_v \to \mathbb{C}$

The Seiberg-Witten Solution

- Key idea: prepotential and special coords should be encoded as a family of algebraic curves, whose global structure is uniquely determined by known constraints.

The Seiberg-Witten Solution

- Key idea: prepotential and special coords should be encoded as a family of algebraic curves, whose global structure is uniquely determined by known constraints.

- SW ansatz: away from some divisor $D \subseteq \mathcal{M}_c$, there is a family $\Sigma \to \mathcal{M}_c \setminus D$ of genus r curves and a 1-form λ on Σ s.t. for any symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r \in H_1(\Sigma_v, \mathbb{Z})$, we can take

$$a_i(v) = \oint_{\alpha_i} \lambda \,, \qquad \frac{d\mathcal{F}}{da_i}(v) = \oint_{\beta_i} \lambda \,.$$

- After the work of SW in the $SU(2)$ case, this ansatz was successfully applied to most SYM and SQCD theories by other groups of authors.

The Seiberg-Witten Solution

<u>Ex</u> SU(2) SYM $\rightsquigarrow \mathcal{M}_C(\mathbb{R}^4) \simeq \mathbb{C}_v$

$$\Sigma = \{ y^2 = z - 2v + z^{-1} \} \subseteq T^*\mathbb{C}^\times_z \times \mathbb{C}_v, \quad \lambda = \frac{y\,dz}{z},$$

$D = \{\pm 1\}$ = pts where Σ_v is singular

The Seiberg-Witten Solution

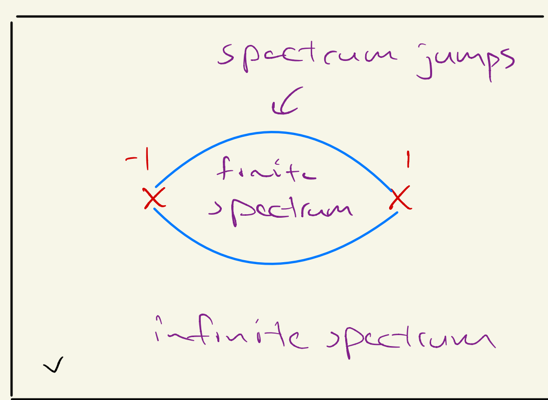
<u>Ex</u> $SU(2)$ SYM $\rightsquigarrow \mathcal{M}_c(\mathbb{R}^4) \simeq \mathbb{C}_v$

$$\Sigma = \{ y^2 = z - 2v + z^{-1} \} \subseteq T^*\mathbb{C}^\times_z \times \mathbb{C}_v, \quad \lambda = \frac{y\,dz}{z},$$

$D = \{\pm 1\}$ = pts where Σ_v is singular

Aside: SW identify $H_1(\Sigma_v, \mathbb{Z})$ w/ the electromagnetic charge lattice Γ_{EM} of the theory $\mathcal{Q}$.

- We can also associate to $\mathcal{Q}$ its Hilbert space $\mathcal{H}$, which is graded by Γ_{EM}. For each $\gamma \in \Gamma_{EM}$, there is a subspace $\mathcal{H}^{BPS}_\gamma \subseteq \mathcal{H}_\gamma$ of BPS states.
- The BPS spectrum is the set of γ with $\mathcal{H}^{BPS}_\gamma \neq 0$.
- SW observed that the spectrum is piecewise-constant in $\mathcal{M}_c$, but could only compute it in small cases.



$SU(2)$ SYM

Compactification on S^1

- The meaning of the curves Σ_u was clarified in subsequent '96 work of Seiberg-Witten.
- They studied $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$, which projects to $\mathcal{M}_c(\mathbb{R}^4)$ and has a degeneration $\mathcal{M}_c(\mathbb{R}^3 \times S^1) \rightsquigarrow \mathcal{M}_c(\mathbb{R}^3)$ over $\mathcal{M}_c(\mathbb{R}^4)$.
- A new feature is these are hyperkahler: they admit a $\mathbb{P}^1$-family of holomorphic symplectic structures satisfying an integrability condition.

Compactification on S^1

- The meaning of the curves Σ_v was clarified in subsequent '96 work of Seiberg-Witten.
- They studied $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$, which projects to $\mathcal{M}_c(\mathbb{R}^4)$ and has a degeneration $\mathcal{M}_c(\mathbb{R}^3 \times S^1) \rightsquigarrow \mathcal{M}_c(\mathbb{R}^3)$ over $\mathcal{M}_c(\mathbb{R}^4)$.
- A new feature is these are hyperkahler: they admit a $\mathbb{P}^1$-family of holomorphic symplectic structures satisfying an integrability condition.
- For $\mathcal{M}_c(\mathbb{R}^3)$ all these structures are isomorphic.
- For $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$ they are not, but in a distinguished complex structure SW (together w/ Donagi-Witten) show it is $\mathrm{Jac}(\Sigma)$, the Jacobian bundle of the SW curves (or a relative of this).

Integrable Systems

- Donagi-Witten further explained that the projection

$$\mathcal{M}_c(\mathbb{R}^3 \times S^1) \simeq \mathrm{Jac}(\Sigma) \longrightarrow \mathcal{M}_c(\mathbb{R}^4)$$

is an integrable system, i.e. it has Lagrangian fibers.

Ex The periodic Toda system ('67) has phase space

$$X_n = \left\{ A(z) = \begin{bmatrix} b_1 & 1 & 0 & \cdots & a_0 z^{-1} \\ a_1 & b_2 & 1 & & \\ 0 & a_2 & & \ddots & 1 \\ \vdots & & & & \\ z & & & a_{n-1} & b_n \end{bmatrix} \quad \text{s.t } \prod a_i = 1, \ \sum b_i = 0 \right\} \subseteq \widehat{\mathfrak{sl}}_n,$$

Hamiltonians the invariants $\mathrm{tr}\, A^2, \ldots, \mathrm{tr}\, A^n$, and symplectic structure obtained by identifying X_n w/ a coadjoint orbit in the Borel of $\widehat{\mathfrak{sl}}_n$.

Integrable Systems

- Donagi-Witten further explained that the projection

$$\mathcal{M}_c(\mathbb{R}^3 \times S^1) \simeq \mathrm{Jac}(\Sigma) \longrightarrow \mathcal{M}_c(\mathbb{R}^4)$$

is an integrable system, i.e. it has Lagrangian fibers.

<u>Ex</u> The periodic Toda system ('67) has phase space

$$X_n = \left\{ A(z) = \begin{bmatrix} b_1 & 1 & 0 & \cdots & a_0 z^{-1} \\ a_1 & b_2 & 1 & & \\ 0 & a_2 & \ddots & & \\ \vdots & & & \ddots & 1 \\ z & & & a_{n-1} & b_n \end{bmatrix} \text{ s.t } \prod a_i = 1, \ \sum b_i = 0 \right\} \subseteq \widehat{\mathfrak{sl}}_n,$$

Hamiltonians the invariants $\mathrm{tr}\, A^2, \ldots, \mathrm{tr}\, A^n$, and symplectic structure obtained by identifying X_n w/ a coadjoint orbit in the Borel of $\widehat{\mathfrak{sl}}_n$.

- The $\mathrm{tr}\, A^k$ are independent and Poisson-commute.
- Their level sets are thus Lagrangian, and labeled by the spectral curves $\det(A(z) - y\,\mathrm{Id})$.
- Remarkably, these are exactly the $SU(n)$ SYM SW curves.

Integrable Systems

- Another key class of examples were given by Hitchin '87:
- C algebraic curve $\rightsquigarrow$ moduli space $\mathcal{M}_{Higgs}$ of Higgs bundles, pairs of rk n vector bundle $V \to C$ and $\varphi : V \to V \otimes \omega$ satisfying a stability condition
- We have a map $\mathcal{M}_{Higgs} \to B = \bigoplus_{k=1}^{n} H^0(\omega^k)$ given by $(V, \varphi) \mapsto (\operatorname{tr}\varphi, \dots, \operatorname{tr}\varphi^n)$.
- Fibers = Jacobians of spectral curves $\{\det(\varphi - \lambda \cdot \mathrm{Id}) = 0\} \subseteq T^*C$
- Moreover, $\mathcal{M}_{Higgs}$ has a holomorphic symplectic structure for which these fibers are Lagrangian.

Integrable Systems

- Another key class of examples were given by Hitchin '87:
- C algebraic curve $\rightsquigarrow$ moduli space $\mathcal{M}_{Higgs}$ of Higgs bundles, pairs of rk n vector bundle $V \to C$ and $\varphi: V \to V \otimes \omega$ satisfying a stability condition
- We have a map $\mathcal{M}_{Higgs} \longrightarrow B = \bigoplus_{k=1}^{n} H^0(\omega^k)$ given by $(V, \varphi) \mapsto (\operatorname{tr}\varphi, \dots, \operatorname{tr}\varphi^n)$.
- Fibers = Jacobians of spectral curves $\{\det(\varphi - \lambda \cdot \mathrm{Id}) = 0\} \subseteq T^*C$
- Moreover, $\mathcal{M}_{Higgs}$ has a holomorphic symplectic structure for which these fibers are Lagrangian.
- Hitchin defines for other groups as well, and others (Markman, Biquard-Boalch) extended to allow singularities in φ.
- By now many (most? all?) SW systems have come to be understood as Hitchin systems of some kind.

Integrable Systems

- The Hitchin construction also gives a description of the hyperkahler structure on $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$ when it can be identified w/ some $\mathcal{M}_{Higgs}$.
- Namely, at $0, \infty \in \mathbb{P}^1$ we have the above complex structure, while the other complex structures are all isomorphic and related to local systems on C.
- For nonsingular φ and group G, this is the character variety $\{\pi_1 C \longrightarrow G\}/G$.

Integrable Systems

- The Hitchin construction also gives a description of the hyperkahler structure on $\mathcal{M}_C(\mathbb{R}^3 \times S^1)$ when it can be identified w/ some $\mathcal{M}_{Higgs}$.
- Namely, at $0, \infty \in \mathbb{P}^1$ we have the above complex structure, while the other complex structures are all isomorphic and related to local systems on C.
- For nonsingular φ and group G, this is the character variety $\{\pi_1 C \longrightarrow G\}/G$.
- For the Toda system, where $C = \mathbb{P}^1$, $G = GL_n$, and φ has an "irregular" pole at $0, \infty \in C$, it is
$$\left\{ g \in GL_n,\ F^0_\bullet, F^\infty_\bullet \in \mathcal{F}\ell_n \text{ s.t. } \begin{matrix} F^0_k \cap gF^\infty_k = F^0_{k-1} \\ F^\infty_k \cap gF^\infty_k = F^\infty_{k-1} \end{matrix} \right\} \Big/ GL_n$$
- Note that these complex structures have the structure of an affine variety, but $\mathcal{M}_{Higgs}$ does not.