Coulomb Branches in Geometry and Representation Theory

(From Seiberg-Witten to BZSV Sec. 8)

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Workshop on Relative Langlands Program

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Last Time

- Compact group G, complex rep $V \rightsquigarrow$
  $\rightsquigarrow$ susy gauge theories $T^{4d}_{G,V}$ and $T^{3d}_{G,V}$.
  $\rightsquigarrow$ Coulomb/Higgs branches on $\mathbb{R}^4$, $\mathbb{R}^3 \times S^1$, $\mathbb{R}^3$.

- $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$ and $\mathcal{M}_c(\mathbb{R}^3)$ are hyperkahler.

- In a distinguished cx structure, $\mathcal{M}_c(\mathbb{R}^3 \times S^1) \longrightarrow \mathcal{M}_c(\mathbb{R}^4)$ is a Lagrangian fibration $\mathrm{Jac}(\Sigma) \longrightarrow \mathbb{C}^r$ for some family of algebraic curves $\Sigma \to \mathbb{C}^r$.

- $\mathcal{M}_c(\mathbb{R}^3)$ has unique cx structure up to isomorphism, is a degeneration of $\mathcal{M}_c(\mathbb{R}^3 \times S^1)$ as $S^1 \rightsquigarrow pt$.

- Seiberg-Witten curves for 4d $\mathcal{N}=2$ SYM
  = spectral curves for affine Toda systems

Integrable Systems

- Another key class of examples were given by Hitchin '87:
- C algebraic curve $\rightsquigarrow$ moduli space $\mathcal{M}_{Higgs}$ of Higgs bundles, pairs of rk n vector bundle $V \to C$ and $\varphi: V \to V \otimes \omega$ satisfying a stability condition
- Map $\mathcal{M}_{Higgs} \longrightarrow \bigoplus_{k=1}^{n} H^0(\omega^k)$, $(V,\varphi) \mapsto (\operatorname{tr}\varphi, \dots, \operatorname{tr}\varphi^n)$.
- Fibers = Jacobians of spectral curves $\{\det(\varphi - \lambda \cdot \mathrm{Id}) = 0\} \subseteq T^*C$
- $\mathcal{M}_{Higgs}$ has a holomorphic symplectic structure such that these fibers are Lagrangian.

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- Fibers = Jacobians of spectral curves $\{\det(\varphi - \lambda \cdot \mathrm{Id}) = 0\} \subseteq T^*C$
- $\mathcal{M}_{Higgs}$ has a holomorphic symplectic structure such that these fibers are Lagrangian.
- Construction extends to $G \neq U(n)$ (Hitchin) and φ with singularities (Markman, Biquard-Boalch).
- Periodic Toda can be interpreted as a singular Hitchin system on $\mathbb{P}^1$, and Donagi-Witten '95 showed that $\mathcal{M}_C(T_{SU(n)}, \mathfrak{sl}_n, \mathbb{R}^3 \times S^1)$ is a singular Hitchin system on an elliptic curve.

Integrable Systems

- The Hitchin construction also gives a description of the hyperkahler structure on $\mathcal{M}_C(\mathbb{R}^3 \times S^1)$ when it can be identified w/ some $\mathcal{M}_{Higgs}$.
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- Namely, at $0, \infty \in \mathbb{P}^1$ we have the above complex structure, while the other complex structures are all isomorphic and related to local systems on C.
- For a group G and nonsingular φ, they are the character variety $\{\pi_1 C \longrightarrow G_{\mathbb{C}}\}/G_{\mathbb{C}}$.
- For periodic Toda, where $C = \mathbb{P}^1$, $G = U(n)$, and φ has certain singularities at $0, \infty \in C$, they are

$$\left\{ g \in GL_n,\ F^0_\bullet, F^\infty_\bullet \in \mathcal{F}\ell_n \text{ s.t. } \begin{array}{l} F^0_k \cap g F^\infty_k = F^0_{k-1} \\ F^\infty_k \cap g F^\infty_k = F^\infty_{k-1} \end{array} \right\} / GL_n$$

- Note that these complex structures have the structure of an affine variety, but $\mathcal{M}_{Higgs}$ does not.

3d Mirror Symmetry

- A remarkable duality between Higgs and Coulomb branches on $\mathbb{R}^3$ was found by Intriligator-Seiberg '96 and further developed by Hanany-Witten '96.

- I-S found that for certain 3d $\mathcal{N}=4$ theories T, there exists a 3d mirror $T^\vee$ such that

$$\mathcal{M}_C(T, \mathbb{R}^3) \simeq \mathcal{M}_H(T^\vee, \mathbb{R}^3) \quad \text{and} \quad \mathcal{M}_H(T, \mathbb{R}^3) \simeq \mathcal{M}_C(T^\vee, \mathbb{R}^3)$$

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- H-W gave a string-theoretic framework for producing many more examples of such pairs.

- In particular, for 3d $\mathcal{N}=4$ theories the relation between Coulomb and Higgs branches is (sometimes) symmetric.

- Aspects of this were rediscovered as the symplectic duality of Braden-Licata-Proudfoot-Webster '07.

3d Mirror Symmetry

Ex (I-S) $V = \mathbb{C}^n$, $G = T(SU(n)) \simeq U(1)^{n-1} \rightsquigarrow V = \mathbb{C}^n$, $G = Z(U(n)) \simeq U(1)$

- This is a simple example of a quiver gauge theory.
- Quiver w/ vertices labeled $n_1, \dots, n_K$, some "circled" and some "boxed"

  $\rightsquigarrow G = \prod_{\text{circled } i} U(n_i)$, $V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$
- Above example: $\boxed{1} \to \overbrace{1 \to \cdots \to 1}^{n-1} \to \boxed{1} \rightsquigarrow 1 \leftarrow \boxed{n}$

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Ex (H-W)

$$\begin{tikzcd}
 & & & \boxed{1} \arrow[d] & & \boxed{1} & & & \\
(1) \arrow[r] & (2) \arrow[r] & \cdots \arrow[r] & (k) \arrow[r] & \cdots \arrow[r] & (k) \arrow[r] \arrow[u] & \cdots \arrow[r] & (2) \arrow[r] & (1)
\end{tikzcd}$$

$$\underbrace{(k) \to \cdots \to (k)}_{m-2k+1}$$

$\rightsquigarrow$ $(k) \leftarrow \boxed{m}$ U(k) SQCD w/ m flavors

Ex (Gaiotto-Witten '07)

$(1) \to (2) \to \cdots \to (k\text{-}1) \to \boxed{k}$ is self-mirror.

3d Mirror Symmetry

- While Coulomb branches are a priori difficult to compute ("quantum corrections"), Higgs branches are easy to compute.
- $G \curvearrowright V \rightsquigarrow$ Hamiltonian action $G \curvearrowright T^*V \simeq V \oplus V^\vee$ w/ moment map $\mu : T^*V \to \mathfrak{g}^*$, $\langle X, \mu(v, \xi)\rangle = \langle Xv, \xi \rangle$
- We then have $\mathcal{M}_H(T_{G,V}) := T^*V /\!/\!/_0 G \simeq \mu^{-1}(0) /\!/ G_{\mathbb{C}}$

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<u>Ex</u> $V = \mathbb{C}^n$, $G = T(SU(n)) \Rightarrow \mu(v,\xi) = (\ldots, v_i\xi_i - v_{i+1}\xi_{i+1}, \ldots)$

$$\Rightarrow \mu^{-1}(0) = \{v_1\xi_1 = \cdots = v_n\xi_n\}$$

- $\mathbb{C}[\mu^{-1}(0)]^G = \mathbb{C}[x,y,z]/\langle xy - z^n\rangle$ for $x = v_1 \cdots v_n$, $y = \xi_1 \cdots \xi_n$, $z = x_1\xi_1 \Rightarrow \mathcal{M}_H \simeq \mathbb{C}^2/\mathbb{Z}_n$

<u>Ex</u> $V = \mathbb{C}^n$, $G = Z(U(n)) \Rightarrow \mu(v,\xi) = \xi(v)$

- $(v,\xi) \mapsto v \otimes \xi \in \mathrm{End}(V)$ identifies $\mathcal{M}_H \simeq \mathcal{O}_{min} \subseteq \mathcal{N}_{A_{n-1}}$

- First case: $\dim \mathcal{M}_H = 2$, $\dim \mathcal{M}_C = 2n-2$

Second case: $\dim \mathcal{M}_H = 2n-2$, $\dim \mathcal{M}_C = 2$

Hitchin Systems $\subseteq$ Coulomb Branches

- In '97, Witten adapted the H-W framework to derive the S-W systems of all 4d $\mathcal{N}=2$ A_n or $A_n^{(1)}$ quiver gauge theories, identifying them as singular Hitchin systems on $\mathbb{P}^1$ or an elliptic curve.
- Crucial insight: such a theory can be interpreted as a compactification of a non-Lagrangian 6d susy QFT $\mathcal{T}_\Gamma^{6d}$ associated to an ADE type Γ and constructed via string theory.

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- Crucial insight: such a theory can be interpreted as a compactification of a non-Lagrangian 6d susy QFT T^{6d}_Γ associated to an ADE type Γ and constructed via string theory.
- Gaiotto '09 extended this, showing a large class of other gauge theories could be approached this way using more general curves.
- Moreover, any ADE Hitchin system of any curve C is the S-W system of the 4d $\mathcal{N}=2$ theory $T^{4d}_{C,\Gamma}$ obtained by compactifying T^{6d}_Γ on C.

Clusters and Lines

- Fock-Goncharov '03: regular functions on the SL_n-character variety of a punctured curve C is a cluster algebra, an algebra $\mathcal{A}_Q$ with (partial) canonical basis associated to a quiver Q (Fomin-Zelevinsky '01).

<u>Ex</u> $\mathcal{A}_{A_2} = \mathbb{C}[x_1, \dots, x_5] / \langle x_{i-1} x_{i+1} = x_i + 1 \rangle_{1 \le i \le 5}$ has a basis $\{ x_i^m x_{i+1}^n \}_{1 \le i \le 5,\, m,n \in \mathbb{N}}$ of cluster monomials, and is functions on a (decorated) SL_2-character variety of $\mathbb{P}^1 \setminus \{pt\}$.

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- Gaiotto-Moore-Neitzke '10: as functions on $\mathcal{M}_C(T_{C, A_{n-1}}, \mathbb{R}^3 \times S^1)$ these canonical basis elements correspond to vacuum expectation values of line defects wrapping S^1.

Clusters and Lines

- Kapustin-Saulina '07 had previously suggested vev's of Wilson-'t Hooft lines $L_{\lambda,\mu}$ give a basis for functions on $\mathcal{M}_c(T_{G,V}, \mathbb{R}^3 \times S^1)$.

- Here λ is a coweight, μ is a weight, and

$$\langle L_{\lambda,\mu} \rangle_v \approx \int_{\mathcal{F}_{v,\lambda}} \mathrm{tr}_{V_\mu} \mathrm{Hol}_{\{m\}\times S^1}(A)\, e^{-iS(\phi)/\hbar} \text{"}d\phi\text{"}$$

fields asymptotic to v w/ λ-monopole singularity along $\{m\}\times S^1$

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- More generally, the analysis of GMN '10 suggests that for any 4d $\mathcal{N}=2$ theory T,

$$\mathcal{M}_c(T, \mathbb{R}^3\times S^1) \simeq \mathrm{Spec}\, A_{Q(T)}$$

where we consider the generic complex structure on the left and $Q(T)$ is the BPS quiver of T.

Clusters and Lines

- Recall the charge lattice $\Gamma_{EM}(v)$ of $\mathcal{T}$ at $v \in \mathcal{M}_C(\mathcal{T}, \mathbb{R}^4)$, which has a skew-symmetric pairing and gives a grading for the Hilbert space $\mathcal{H}(v)$ of $\mathcal{T}$ at v.
- The BPS spectrum at v is the set of $\gamma \in \Gamma_{EM}(v)$ such that $\mathcal{H}^{BPS}_{\gamma}(v) \neq 0$.

Clusters and Lines

- Recall the charge lattice $\Gamma_{EM}(v)$ of T at $v \in \mathcal{M}_C(T, \mathbb{R}^4)$, which has a skew-symmetric pairing and gives a grading for the Hilbert space $\mathcal{H}(v)$ of T at v.
- The BPS spectrum at v is the set of $\gamma \in \Gamma_{EM}(v)$ such that $\mathcal{H}^{BPS}_{\gamma}(v) \neq 0$.
- Then $\begin{matrix}\text{BPS}\\ \text{quiver}\end{matrix} : \begin{matrix}\text{BPS}\\ \text{spectrum}\end{matrix} :: \begin{matrix}\text{Dynkin}\\ \text{diagram}\end{matrix} : \begin{matrix}\text{root}\\ \text{system}\end{matrix}$
- This quiver depends on the choices made, but the resulting cluster algebra does not.
- Cecotti '12: explicit recipe for $Q(T^{4d}_{G,V})$.

Ex:

$G = SU(4)$, $V = V_{\omega_1 + 2\omega_3}$ $\rightsquigarrow$ $Q(T_{G,V}) =$