Coulomb Branches in Geometry and Representation Theory

(From Seiberg-Witten to BZSV Sec. 8)

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Workshop on Relative Langlands Program

National University of Singapore, 1/7/26-1/9/26

Last Time

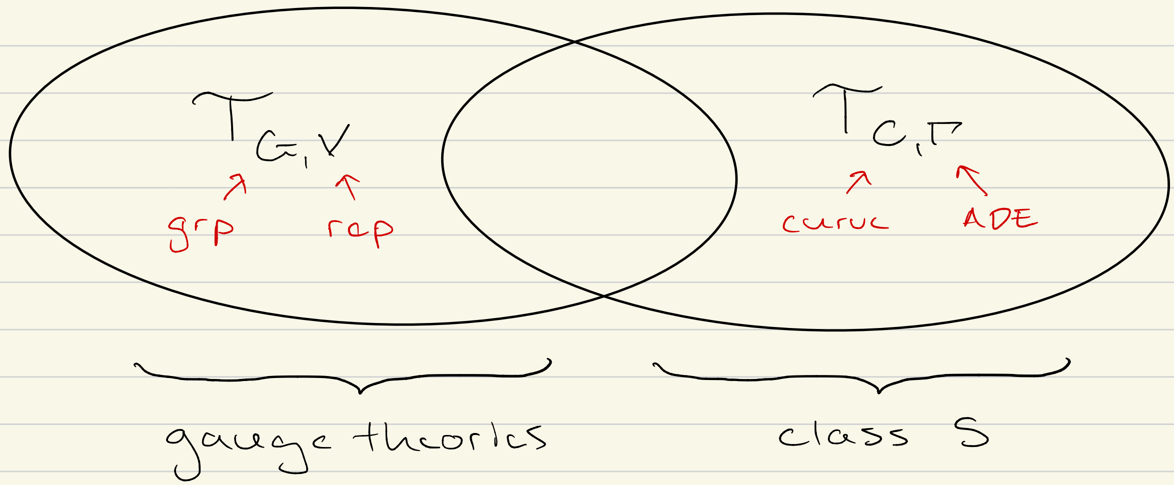
Coulomb branches

- $4d\ \mathcal{N}=2$ theory : $\mathcal{M}_C(\mathbb{R}^3\times S^1) \longrightarrow \mathcal{M}_C(\mathbb{R}^4)$

$\quad\quad \Big\downarrow S^1 \to pt \quad\quad\quad\quad\quad \Big\downarrow \quad \nearrow$

$3d\ \mathcal{N}=4$ theory : $\mathcal{M}_C(\mathbb{R}^3)$

- Two main classes of $4d\ \mathcal{N}=2$ theory:



- Mathematical realizations of Coulomb branches:

1) $\mathcal{M}_C(T_{C,\Gamma}, \mathbb{R}^3\times S^1) =$ Hitchin moduli space of C,Γ

2) T is 3d mirror to a gauge theory $T_{G,V}$

$$\Rightarrow \mathcal{M}_C(T, \mathbb{R}^3) \simeq \mathcal{M}_H(T_{G,V}) \simeq T^*V /\!/\!/_0\, G$$

Clusters and Lines

- Fock-Goncharov '03: regular functions on the SL_n-character variety of a punctured curve C is a cluster algebra, an algebra $\mathcal{A}_Q$ with (partial) canonical basis associated to a quiver Q (Fomin-Zelevinsky '01).

Ex $\mathcal{A}_{A_2} = \mathbb{C}[x_1, \ldots, x_5] / \langle x_{i-1} x_{i+1} = x_i + 1 \rangle_{1 \le i \le 5}$ has a basis $\{x_i^m x_{i+1}^n\}_{1 \le i \le 5,\, m,n \in \mathbb{N}}$ of cluster monomials, and is functions on a (decorated) SL_2-character variety of $\mathbb{P}^1 \setminus \{pt\}$.

- Gaiotto-Moore-Neitzke '10: as functions on $\mathcal{M}_C(T_{C, A_{n-1}}, \mathbb{R}^3 \times S^1)$ these canonical basis elements correspond to vacuum expectation values of irreducible line defects wrapping S^1.

Clusters and Lines

- Kapustin-Saulina '07 had previously suggested vev's of Wilson-'t Hooft lines $L_{\lambda,\mu}$ give a basis for functions on $\mathcal{M}_c(T_{G,V}, \mathbb{R}^3 \times S^1)$.

- Here λ is a coweight, μ is a weight, and

$$\langle L_{\lambda,\mu} \rangle_v \approx \int_{\mathcal{F}_{v,\lambda}} \mathrm{tr}_{V_\mu} \mathrm{Hol}_{\{m\}\times S^1}(A) \, e^{-iS(\phi)/\hbar} \, \text{“}d\phi\text{”}$$

↖ fields asymptotic to v w/ λ-monopole singularity along $\{m\}\times S^1$

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- But the arguments of GMN are more general, suggesting another Coulomb realization:

3) $\mathcal{M}_C(T, \mathbb{R}^3 \times S^1) \approx \mathrm{Spec}\, \mathcal{A}_{Q(T)}$, where we take the generic complex structure on the left and $Q(T)$ is the BPS quiver of T.

Clusters and Lines

- Recall the charge lattice $\Gamma_{EM}(v)$ of $\mathcal{T}$ at $v \in \mathcal{M}_c(\mathcal{T}, \mathbb{R}^4)$, which has a skew-symmetric pairing and gives a grading for the Hilbert space $\mathcal{H}(v)$ of $\mathcal{T}$ at v.
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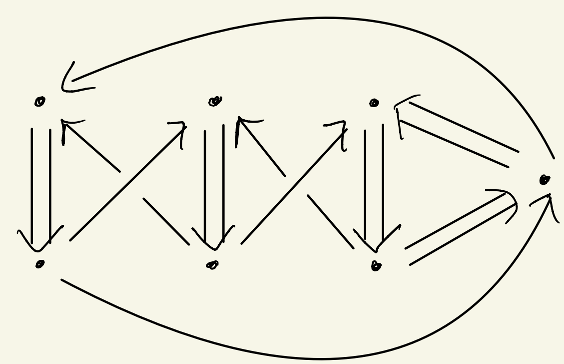
- The BPS spectrum at v is the set of $\gamma \in \Gamma_{EM}(v)$ such that $\mathcal{H}^{BPS}_{\gamma}(v) \neq 0$.
- Then $\begin{matrix}\text{BPS} \\ \text{quiver}\end{matrix} : \begin{matrix}\text{BPS} \\ \text{spectrum}\end{matrix} :: \begin{matrix}\text{Dynkin} \\ \text{diagram}\end{matrix} : \begin{matrix}\text{root} \\ \text{system}\end{matrix}$
- This quiver depends on the choices made, but the resulting cluster algebra does not.
- Cecotti '12: explicit recipe for $Q(\mathcal{T}_{G,V})$.

<u>Ex:</u> $G = SU(4)$, $V = V_{\omega_1 + 2\omega_3}$ $\rightsquigarrow$ $Q(\mathcal{T}_{G,V}) =$



The BFN Construction

- Braverman-Finkelberg-Nakajima '15 gave a general realization of $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3)$ and $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3 \times S^1)$, building on calculations of Cremonesi-Hanany-Zaffaroni '14.

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- Key idea: $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3)$ should be the Hilbert space to S^2 assigned by a topological twist of $T_{G,V}$, which should in turn be a linearization of the space of maps $S^2 \to V/G$

- We can model S^2 in algebraic geometry as $\mathbb{B} = D \cup_{D^\times} D$, the "bubble" of formal disks, and linearize by taking Borel-Moore homology.

The BFN Construction

- Letting $\mathcal{O} = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$, we have

$$\mathrm{Maps}(\mathbb{B}, V/G) \simeq V_{\mathcal{O}}/G_{\mathcal{O}} \times_{V_K/G_K} V_{\mathcal{O}}/G_{\mathcal{O}} \simeq R_{G,V}/G_{\mathcal{O}},$$

where $R_{G,V} := \left\{ [g] \in Gr_G,\ s \in V_{\mathcal{O}} \cap_{V_K} gV_{\mathcal{O}} \right\} / G_{\mathcal{O}}$

- Then the BFN Coulomb realization is:

4) $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3) \simeq H_{\bullet}^{G_{\mathcal{O}}}(R_{G,V})$ and in its generic complex structure $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3 \times S^1) \simeq K^{G_{\mathcal{O}}}(R_{G,V})$.

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- Setting $Y = V_{\mathcal{O}}/G_{\mathcal{O}}$, $Z = V_K/G_K$, the diagram

$$(Y \times_Z Y) \times (Y \times_Z Y) \xleftarrow{\Delta_Y'} Y \times_Z Y \times_Z Y \xrightarrow{\pi_{13}} Y \times_Z Y$$

induces a product on both linearizations.

- These turn out to be commutative, and are quantized by adding loop rotation equivariance.

The BFN Construction

- Let's check $\mathcal{M}_C(T_{Z(GL_n)}, \mathbb{C}^n) \simeq \mathcal{M}_H(T_{T(SL_n)}, \mathbb{C}^n) \simeq \mathbb{C}^2/\mathbb{Z}_n$.
- Torus $T \curvearrowright V \Rightarrow H_\bullet^T(V)$ has the following features:

1) $H_\bullet^T(V)$ freely generated over $H_T^\bullet(pt) \simeq \mathbb{C}[t]$ by $[V]$.
2) For $W \subseteq V$, $[W] = e(V/W)[V]$, where the Euler class $e(V/W) \in H_T^\bullet(pt)$ is the product of the weights of V/W.
3) We have $H_T^\bullet(pt)$-linear maps $i_* : H_\bullet^T(W) \leftrightarrows H_\bullet^T(V) : i^!$ given by $i_*[W] = [W]$ and $i^![V] = [W]$
4) We have $[W] \underset{V}{\cap} [W'] = e(V/W+W')[W \cap W']$.

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- Recall that $Gr_T \simeq \{[t^\lambda]\}_{\lambda \in P^\vee}$, hence $R := R_{T,V}$ is $\bigsqcup_{\lambda \in P^\vee} R^\lambda$, where $R^\lambda = V_O \underset{V_K}{\cap} t^\lambda V_O$.

- $H_\bullet^T(R)$ generated as $H_T^\bullet(pt)$-algebra by $\{m_\lambda = [R^\lambda]\}_{\lambda \in P^\vee}$, with identity m_0.

The BFN Construction

- Given $\lambda_1, \lambda_2 \in P^\vee$, write $V^0 = V_\mathcal{O}$, $V^1 = t^{\lambda_1} V_\mathcal{O}$, $V^2 = t^{\lambda_1+\lambda_2} V_\mathcal{O}$, and $V^{01} = V^0 \cap V^1$, etc...

- Writing $V^1 \xleftarrow{i_1} V^{012} \xrightarrow{i_{02}} V^{02} = R^{\lambda_1+\lambda_2}$, we then have

$$m_{\lambda_1} m_{\lambda_2} = i_{02*} i_1^! \left([V^{01}] \underset{V_1}{\cap} [V^{12}] \right)$$

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- Now take $V = \mathbb{C}^n$, $T = Z(GL_n)$, so $P^\vee \simeq \mathbb{Z}$, $H_T^\bullet(pt) \simeq \mathbb{C}[\xi]$.

$\underline{\lambda_1 = 1, \lambda_2 = -1:}$ $V^0 = V^2 = V_\mathcal{O}$, $V^1 = tV_\mathcal{O} \implies tV_\mathcal{O} \xleftarrow[\sim]{i_1} tV_\mathcal{O} \xrightarrow{i_{02}} V_\mathcal{O}$

$$\implies m_1 m_{-1} = i_{02*} i_1^! [tV_\mathcal{O}] = e(V_\mathcal{O}/tV_\mathcal{O})[V_\mathcal{O}] = \xi^n m_0$$

$\underline{\lambda_1 = -1, \lambda_2 = 1:}$ $V^0 = V^2 = V_\mathcal{O}$, $V^1 = t^{-1} V_\mathcal{O} \implies t^{-1} V_\mathcal{O} \xleftarrow{i_1} V_\mathcal{O} \xrightarrow[\sim]{i_{02}} V_\mathcal{O}$

$$\implies m_{-1} m_1 = i_{02*} i_1^! e(t^{-1}V_\mathcal{O}/V_\mathcal{O})[t^{-1}V_\mathcal{O}] = \xi^n m_0$$

- One similarly checks $m_1^k = m_k$, $m_{-1}^k = m_{-k}$, hence

$$H^T(R) \simeq \mathbb{C}[m_1, m_{-1}, \xi]/\langle m_1 m_{-1} - \xi^n \rangle \simeq \mathbb{C}[\mathbb{C}^2/\mathbb{Z}_n]$$

The BFN Construction

- Similarly, as a $K^T(pt) = \mathbb{C}[T]$-algebra $K^T(\mathcal{R})$ is generated by $\{x_\lambda = [\mathcal{O}_\lambda := \mathcal{O}_{\mathcal{R}^\lambda}]\}_{\lambda \in P^\vee}$, and

$$x_{\lambda_1} x_{\lambda_2} = [\mathcal{O}_{\lambda_1} * \mathcal{O}_{\lambda_2} = i_{02*} i_1^*(\mathcal{O}_{V^{01}} \otimes_{\mathcal{O}_{V^1}} \mathcal{O}_{V^{12}})]$$

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- Now take $V = \mathbb{C}$, $T = GL_1$, so $K^T(pt) = \mathbb{C}[z^{\pm 1}]$, where z is the usual generator w/ a sign. Write $\mathcal{F} \mapsto \mathcal{F}\{1\}$ for the equivariant shift, so e.g. $[\mathcal{F}\{1\}] = -z[\mathcal{F}]$.

$\underline{\lambda_1 = 1, \lambda_2 = -1:}$ $V^0 = V^2 = V_\mathcal{O}$, $V^1 = tV_\mathcal{O}$ $\Rightarrow$ $tV_\mathcal{O} \xleftarrow[\sim]{i_1} tV_\mathcal{O} \xhookrightarrow{i_{02}} V_\mathcal{O}$

$\Rightarrow \mathcal{O}_1 * \mathcal{O}_{-1} \simeq i_{02*} i_1^* \mathcal{O}_{tV_\mathcal{O}} \simeq \mathcal{O}_{V_\mathcal{O}}\{1\} \longrightarrow \mathcal{O}_{V_\mathcal{O}}$

$\Rightarrow x_1 x_{-1} = x_0 + z x_0$

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$\Rightarrow \mathcal{O}_{-1} * \mathcal{O}_1 \simeq i_{02*} i_1^* (\mathcal{O}_{V_\mathcal{O}} \otimes_{\mathcal{O}_{t^{-1}V_\mathcal{O}}} \mathcal{O}_{V_\mathcal{O}} \simeq \mathcal{O}_{t^{-1}V_\mathcal{O}}\{1\} \to \mathcal{O}_{t^{-1}V_\mathcal{O}}) \simeq \mathcal{O}_{V_\mathcal{O}}\{1\} \xrightarrow{0} \mathcal{O}_{V_\mathcal{O}}$

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- Again we could check $x_1^k = x_k$, $x_{-1}^k = x_{-k}$, hence $K^T(R) \simeq \mathbb{C}[x_1, x_{-1}, z^{\pm 1}]/\langle x_1 x_{-1} = 1+z \rangle \simeq A_{\bullet \to \square}$, the cluster algebra of the A_1-quiver w/ a frozen variable.

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- In gauge theory terms, $Coh^{G_\mathcal{O}}(R_{G,V})$ provides a realization of the category of line defects in a holomorphic-topological twist of $T_{G,V}$.

- Wilson-'t Hooft lines are realized by irreducible Koszul-perverse sheaves (Cautis-W. '23), which generalize the coherent IC sheaves studied by Bezrukavnikov-Finkelberg-Mirkovic '03.

- They are known to categorify the expected cluster algebra when $G = GL_2$, $V = \mathbb{C}^2$ and $G = GL_n$, $V = 0$ (Cautis-W. '18).

The Plancherel Algebra

- Gaiotto-Witten '08: the 3d $\mathcal{N}=4$ theory $\mathcal{T}^{3d}_{G,V}$ yields a boundary condition for 4d $\mathcal{N}=4$ SYM (i.e. $\mathcal{T}^{4d}_{G,\mathfrak{g}}$).

- This should produce an algebra $A_{G,V}$ in the Satake category $Sh^{G_O}(Gr_G)$, the category of line defects in A-twisted 4d $\mathcal{N}=4$ SYM.

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- BFN '19 realize $A_{G,V}$ in terms of $R_{G,V}$ as follows.

- We can interpret $H^{G_O}_{\bullet}(R_{G,V})$ as the renormalized cohomology of the dualizing sheaf ω_R, a certain shifted pushforward along $R_{G,V} \to pt$.

- But this projection factors through $R_{G,V} \xrightarrow{p} Gr_G$, and $p_*^{ren}\omega_R$ is an algebra in $Sh^{G_O}(Gr_G)$ for the same reason $H^{G_O}_{\bullet}(R)$ is an ordinary algebra.

The Plancherel Algebra

- An alternative realization of $\mathcal{A}_{G,V}$ was given by BDFRT '22 and BZSV '24. In BZSV it is called the Plancherel algebra $\mathbb{PL}_V$ and generalized to the nonlinear context.

- There is a natural action $Sh^{G_O}(Gr_G) \curvearrowright Sh^{G_O}(V_K)$, and $\mathbb{PL}_V$ is defined as the relative endomorphism algebra of the basic object $\omega_{V_O} \in Sh^{G_O}(V_K)$.
  (i.e. $Hom(\mathcal{F}, End^{rel}(\omega_{V_O})) = Hom(\mathcal{F} * \omega_{V_O}, \omega_{V_O})$).

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- The works above show $\mathbb{PL}_V \simeq \mathcal{A}_{G,V}$, and since by construction $H^\bullet \mathcal{A}_{G,V} \simeq H_\bullet^{G_O}(\mathcal{R}_{G,V})$, we obtain another Coulomb branch realization:

5) $\mathcal{M}_C(T_{G,V}, \mathbb{R}^3) \simeq \operatorname{Spec} H^\bullet \mathbb{PL}_V$