TOPICS IN TOPOLOGY: STABLE ∞-CATEGORIES

1. 8/23/21

For the first few lectures we will try to motivate the theory of stable ∞-categories from the point of view of algebra, in particular the theory of derived functors and categories. These in turn are motivated by the following general strategy in mathematics: to study a complicated object, try to build it out of simple parts and then reduce questions about the complicated object to questions about the simple parts.

Consider the following example in the setting of abelian groups. Given $n \in \mathbb{N}$, we have the cyclic group

$$\mathbb{Z}/n\mathbb{Z} := \operatorname{cok}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}),$$

i.e. the cokernel of the map from $\mathbb{Z}$ to itself given by multiplication by n. We can think of this description as explaining how to build the "complicated" abelian group $\mathbb{Z}/n\mathbb{Z}$ out of two copies of the "simple" abelian group $\mathbb{Z}$. (Of course this is a toy example and $\mathbb{Z}/n\mathbb{Z}$ isn't that complicated, but it's more complicated than $\mathbb{Z}$ in so far as you learn ordinary arithmetic before modular arithmetic).

Suppose we want to understand the tensor product $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}$ for some $m \in \mathbb{N}$. Naively, we could try to use the above description of $\mathbb{Z}/n\mathbb{Z}$ together with the trivial identity $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$. That is, we could naively expect that

$$\begin{aligned}
\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} &\cong \mathbb{Z}/m\mathbb{Z} \otimes \operatorname{cok}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) \\
&\cong \operatorname{cok}(\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{n \otimes 1} \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}) \\
&\cong \operatorname{cok}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}),
\end{aligned}$$

and then identify this last line as $\mathbb{Z}/\gcd(m,n)$ using Bezout's identity. And in fact this is actually correct! The key fact that makes this work is that the operation $A \mapsto \mathbb{Z}/m\mathbb{Z} \otimes A$ preserves cokernels of abelian groups.

Now suppose we try to compute $\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, the abelian group of homomorphisms from $\mathbb{Z}/m\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$, the same way. Our naive calculation would be

$$\begin{aligned}
\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) &\cong \operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \operatorname{cok}(\mathbb{Z} \xrightarrow{n} \mathbb{Z})) \\
&\cong \operatorname{cok}(\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \xrightarrow{n} \operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})) \\
&\cong \operatorname{cok}(0 \xrightarrow{n} 0),
\end{aligned}$$

which is of course just zero. But this is obviously wrong: say, when $m = n$ the identity morphism is a nonzero element of $\operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. The key fact that makes this fail is that the operation $A \mapsto \operatorname{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, A)$ does not preserve cokernels of abelian groups.

This distinction reflects a sense in which the operation $A \mapsto \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, A)$ is harder to study than $A \mapsto \mathbb{Z}/m\mathbb{Z} \otimes A$: there is at least one kind of argument available to us in studying the latter which is not available when studying the former.

Note that in the example the role of cokernels is a kind of "blueprint" for how to build a complicated thing out of two (potentially) simpler things. This is a special case of a colimit:

$$\mathrm{cok}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) \cong \mathrm{colim}\left(\begin{array}{ccc} \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} \\ \downarrow & & \\ 0 & & \end{array}\right).$$

A colimit is a more general kind of blueprint for building a compicated thing out of a collection of (potentially) simpler things organized into a diagram (such as the one to the right above). We also have the related notion of limit, which again reads a diagram as a blueprint for building a new object but does so in a dual way. These satisfy dual universal properties, but for now these universal properties aren't what we care about per se — just the fact that both notions give a way of building complicated objects out of simpler ones.

Returning to the example, the moral is that our ability to understand a given operation (e.g. a functor like $\mathbb{Z}/mZ \otimes -$ or $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/m\mathbb{Z}, -)$) is constrained by the types of blueprints it preserves — the more it preserves, the easier it will be to studay. With that in mind, the deep idea of derived categories and functors is the following: to study a functor which doesn't preserve some class of blueprints, try to "upgrade" it so that it does, then study the "upgraded" functor. To discuss this more precisely, let's introduce some terminology.

Definition 1.1. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.

(1) F is right exact if it preserves all finite colimits.
(2) F is left exact if it preserves all finite limits.
(3) F is exact if it is both right and left exact.

This terminology is in turn motivated by the following special case. Recall that a category is abelian if, roughly speaking, it behaves like the category of abelian groups.

Theorem 1.2. *A functor $F : \mathcal{C} \to \mathcal{C}'$ between abelian categories is right exact (resp. left exact) if and only if it preserves finite direct sums and cokernels (resp. kernels).*

Now we can give a slightly more precise version of the "deep idea" above. Suppose $F : \mathcal{C} \to \mathcal{C}'$ is a functor between abelian categories which is left but not right exact (a dual version of the following discussion applies to right but not left exact functors). Then (under mild hypotheses) we can canonically "upgrade" F to an exact functor $R^\bullet F$. BUT we have to "upgrade" the source and target category as well, so that we have a functor

$$R^\bullet F : \mathcal{D}^-(\mathcal{C}) \to \mathcal{D}^-(\mathcal{C}').$$

Here $\mathcal{D}^-(\mathcal{C})$ and $\mathcal{D}^-(\mathcal{C}')$ are called the (bounded below) "derived categories" of $\mathcal{C}$ and $\mathcal{C}'$, and $R^\bullet F$ is called the "total right derived functor" of F.

2. 8/27/21

2.1. **Review.**

2.1.1. *General problem.* Given a functor $F\colon \mathcal{C} \to \mathcal{C}'$ and an object $X \in \mathcal{C}$, how shall we compute $F(X)$?

2.1.2. *General strategy.* Find a way to write

$$X \cong \operatorname*{colim}_{i\in I} X_i \text{ (or } X \simeq \lim_{i\in I} X_i)$$

so that each X_i is simpler than X and so that F perserves $\operatorname*{colim}_{i\in I} X_i$ and $\lim_{i\in I} X_i$, i.e.

$$F(\operatorname*{colim}_{i\in I} X_i) = \operatorname*{colim}_{i\in I} F(X_i),\ F(\lim_{i\in I} X_i) = \lim_{i\in I} F(X_i).$$

In general, functors which preserve many limits or colimits are easier to study.

2.2. **Extended general strategy.** If you can, upgrade F so that it preserves more limits or colimits, hence it becomes easier to study.

2.2.1. *Specific instance.* Given a left (but not right) exact functor, $F\colon \mathcal{C} \to \mathcal{C}'$ of Abelian categories we can upgrade F to an exact "total (right) derived functor" $R^\bullet F\colon D^-(\mathcal{C}) \to D^-(\mathcal{C}')$ between "(bounded below) derived categories".

2.2.2. *Question.* How is $D^-(\mathcal{C})$ related to $\mathcal{C}$ and how is $R^\bullet F$ related to F?

2.2.3. *Rough description of $D^-(\mathcal{C})$.*

(1) For each $n \in \mathbb{Z}$ we have a fully faithful functor $i_n\colon \mathcal{C} \to D^-(\mathcal{C})$ and an essentially surjective functor ("cohomology") $H^n\colon D^-(\mathcal{C}) \to \mathcal{C}$ such that for all $m, n \in \mathbb{Z}$,

$$H^n \circ i_m \cong \begin{cases} \mathrm{id}_{\mathcal{C}} & m = n \\ 0 & m \neq n \end{cases}$$

(2) There is an autoequivalence ("the shift functor") $[1]\colon \mathcal{D}^-(\mathcal{C}) \to \mathcal{D}^-(\mathcal{C}')$ such that $H^n \circ ([1]) \simeq H^{n+1}$ and $[1] \circ i_n \simeq i_{n-1}$.
(3) $\mathcal{D}^-(\mathcal{C})$ has a zero object and $X \simeq 0$ if and only if $H^n(x) \simeq 0$ for all n.

2.2.4. *Analogy.* Passing from $\mathcal{C}$ to $\mathcal{D}^-(\mathcal{C})$ is like passing from $\mathbb{R}$ to $\mathbb{C}$: the latter is more abstract but it's easier to study because it has better formal properties. The cohomologies $H^n(X) \subset \mathcal{C}$ are analogous to the real and imaginary parts of a complex number.

2.2.5. *Warnings.* Given $X \in \mathcal{D}^-(\mathcal{C})$, for all $n \in \mathbb{Z}$ implies $X \simeq 0$. But given $X, Y \in \mathcal{D}^-(\mathcal{C})$, $H^n(X) \simeq H^n(Y)$ for all $n \in \mathbb{Z}$ does not imply $X \simeq Y$.

Later we'll axiomitize the above data and their main properties as saying "$\mathcal{D}^-(\mathcal{C})$ has a t−structure whose heart is $\mathcal{C}$.

2.2.6. *Rough description of $R^\bullet F$.*

(1) We can recover F from

$$\begin{tikzcd} \mathcal{D}^-(\mathcal{C}) \arrow[r, "R^\bullet F"] \arrow[d, "i_0"'] & \mathcal{D}^-(\mathcal{C}') \arrow[d, "H^0"] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{C}' \end{tikzcd}$$

(2) Given $n \in \mathbb{Z}$, we call the functor $R^n F$ defined by

$$\begin{tikzcd} \mathcal{D}^-(\mathcal{C}) \arrow[r, "R^\bullet F"] \arrow[d, "i_0"'] & \mathcal{D}^-(\mathcal{C}') \arrow[d, "H^n"] \\ \mathcal{C} \arrow[r, "R^n F"] & \mathcal{C}' \end{tikzcd}$$

the "nth (right) derived functors of F".

(3) We have $R^n F \simeq 0$ for $n < 0$ ($R^\bullet F$ is "left $t-$exact").

(4) The right derived functors together take short exact sequences in $\mathcal{C}$ to long exact sequences in $\mathcal{C}'$.

3. 8/30/21

Last time: Given a left exact functor $F\colon \mathcal{C} \to \mathcal{C}'$ between 2 abelian categories we described the "total right derived functor"

$$R^\bullet F\colon D^-(\mathcal{C}) \to D^-(\mathcal{C}').$$

- Ultimately $R^\bullet F$ is <u>easier</u> to study than F because it's exact. Aside: last time in our "rough description of $D^-(\mathcal{C})$", part (3) a morphism $f\colon X \to Y$ in $D^-(\mathcal{C})$ is an isomorphism if and only if $H^n(f)\colon H^n(X) \to H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$. Last thing we said: (4) The $\{R^n F\}_{n\in\mathbb{Z}}$ together take short exact sequences in $\mathcal{C}$ to long exact sequences in $\mathcal{C}'$. (Historically, people discovered the $R^n F$'s before they discovered $R^\bullet F$ and $D^-(\mathcal{C})$).

Example (abelian groups)

- Fact (specific to Ab): given $A, B \in Ab$, $R^n\mathrm{Hom}_{Ab}(A,B) \sim 0$ for $n > 1$(and $n < 0$).

- convention: we write Ext^n for $R^n\mathrm{Hom}$

Back to our example: (4) says we have a LES:

$$\begin{tikzcd} & 0 \arrow[d, dash, "\simeq"] & 0 \arrow[d, dash, "\simeq"] & \\ 0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}) \arrow[r, "n"] & \mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}) \arrow[r] & \mathrm{Hom}(\mathbb{Z}/m,\mathbb{Z}/n) \arrow[dll, out=-10, in=170] & \\ & \mathrm{Ext}^1(\mathbb{Z}/m,\mathbb{Z}) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Ext}^1(\mathbb{Z}/m,\mathbb{Z}) \arrow[r] \arrow[d, dash, "\simeq"] & \mathrm{Ext}^1(\mathbb{Z}/m,\mathbb{Z}/n) \arrow[r] & 0 \\ & \mathbb{Z}/m \arrow[r, "n"] & \mathbb{Z}/m & & \end{tikzcd}$$

$\Rightarrow \mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/n) \simeq \mathbb{Z}/\gcd(m,n)$
$\mathbb{Z}/\gcd(m,n) \simeq \mathbb{Z}/\frac{m}{d} \simeq \frac{m}{d}\mathbb{Z}/m$.
More generally, suppose $0 \to A \xrightarrow{f} B \to C \to 0$ is a SES in $\mathcal{C}$.
Then $\{R^nF(C)\}_{n\in\mathbb{Z}}$ is sometimes determined by the data of $\{R^nF(A) \xrightarrow{R^nF(f)} R^nF(B)\}_{n\in\mathbb{Z}}$.
BUT, $R^\bullet F(i_0C)$, hence $\{R^nF(C)\}_{n\in\mathbb{Z}}$, is determined by $R^\bullet F(A) \xrightarrow{R^\bullet F(f)} R^\bullet F(B)$.
$\Rightarrow$ we lose information by passing from $R^\bullet F$ to the $\{R^nF\}_{n\in\mathbb{Z}}$.

4. 9/01/21

Big picture so far: Derived categories "upgrade" abelian categories to have more exact functors between them, which makes them easier to study.

Given a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ between two abelian categories, we discussed

- The total right derived functor $R^\bullet F : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$
- Component right derived functors $R^nF : \mathcal{C} \to \mathcal{C}'$

Theorem 4.1. *Suppose we have a short exact sequence $0 \to A \xrightarrow{\phi} B \to C \to 0$ in $\mathcal{C}$.*

(1) $F(\phi) : F(A) \to F(B)$ is rarely enough to determine $F(C)$
(2) $R^nF(\phi) : R^nF(A) \to R^nF(B)$ is sometimes enough to determine $R^nF(C)$.
(3) $R^\bullet F(i_0\phi) : R^\bullet F(i_0A) \to R^\bullet F(i_0B)$ is always enough to determine $R^\bullet F(i_0C)$.

Recall from last time that we can compute $F(C)$ from $R^\bullet F(i_0C)$, so $R^\bullet F(i_0\phi) : R^\bullet F(i_0A) \to R^\bullet F(i_0B)$ also determines $F(C)$. A naive argument that (3) is true could go as follows:

$$\begin{aligned} R^\bullet F(i_0C) &\cong R^\bullet F(i_0(\mathrm{cok}(A \xrightarrow{\phi} B))) \\ &\cong R^\bullet F(\mathrm{cok}(i_0A \xrightarrow{i_0\phi} i_0B)) \\ &\cong \mathrm{cok}(R^\bullet F(i_0A) \xrightarrow{R^\bullet F(i_0\phi)} R^\bullet F(i_0B)) \end{aligned}$$

The first isomorphism is immediate, and the third isomorphism holds because $R^\bullet F$ is an exact functor, so all we need to examine is the second isomorphism. In particular, we want to know if i_0 preserves cokernels. This "naive" argument actually works, but there is a subtlety that arises.

Important note

There are two versions of the derived category $D^-(\mathcal{C})$.

(1) The derived 1-category $D^-_1(\mathcal{C})$
(2) The derived ∞-category $D^-_\infty(\mathcal{C})$

Rough idea of ∞-categories

Definition 4.2. **Top** is the category whose objects are topological spaces and whose morphisms are continuous maps.

Definition 4.3. An ∞-category $\mathcal{C}$ is a category enriched in **Top**.

This means that an ∞-category $\mathcal{C}$ has

- A set of objects like a normal category
- For each pair of objects $X, Y \in Ob(\mathcal{C})$, a topological space of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- For all $X, Y, Z \in Ob(\mathcal{C})$, a continuous composition map $\circ : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

Definition 4.4. A ∞-functor $F : \mathcal{C} \to \mathcal{C}'$ is a functor enriched in **Top**. Namely, $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is a continuous map for all pairs of objects $X, Y \in Ob(C)$.

A normal 1-category is naturally an ∞-category by giving the each hom-set the discrete topology. Conversely, if $\mathbb{C}$ is an ∞-category, we define a 1-category $h\mathcal{C}$ called the homotopy category. $h\mathbb{C}$ has the same objects as $\mathcal{C}$, and the hom-sets of $h\mathcal{C}$ are given by the connected components of the hom-sets of $\mathcal{C}$.

- $Ob(\mathcal{C}) = Ob(h\mathcal{C})$
- $\mathrm{Hom}_{h\mathcal{C}}(X, Y) := \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$

Given a morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, we can get a morphism in $\mathrm{Hom}_{h\mathcal{C}}(X, Y)$ by taking the connected component which it lies in.

There is an obvious ∞-functor

$$h : \mathcal{C} \to h\mathcal{C}$$

. This functor serves as a bridge between the two versions of the derived category.

$$D_1^-(\mathcal{C}) \cong h(D_\infty^-(\mathcal{C}))$$

Everything we've mentioned so far in our rough description of $D^-(\mathcal{C})$ applies to both versions.

There are also two versions of the functors i_n and H^n, which are related by h as follows:

$$\begin{tikzcd}
& D_\infty^-(\mathcal{C}) \arrow[dd, "h"] \arrow[dr, "H^n"] & \\
\mathcal{C} \arrow[ur, "i^n"] \arrow[dr, "i^n"'] & & \mathcal{C} \\
& D_1^-(\mathcal{C}) \arrow[ur, "H^n"'] &
\end{tikzcd}$$

5. 9/3/2021

The above diagram implies that, given two objects $X, Y \in D_\infty^-(\mathcal{C})$, we have

$$\mathrm{Hom}_{D_1^-(\mathcal{C})}(X, Y) \simeq \pi_0 \mathrm{Hom}_{D_\infty^-(\mathcal{C})}(X, Y).$$

So $D_\infty^-(\mathcal{C})$ contains more information than $D_1^-(\mathcal{C})$.

Recall: Given $F \colon \mathcal{C} \to \mathcal{C}'$ a left exact functor between abelian categories and $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathcal{C}$, then $R^\bullet F(i_0 C)$ is determined by $R^\bullet F(i_0 A) \xrightarrow{R^\bullet F(i_0 \phi)} R^\bullet F(i_0 B)$(whereas $F(C)$ is not determined by $F(A) \xrightarrow{F(\phi)} F(B)$ in general). This is true as stated for derived ∞-categories: $i_0\colon \mathcal{C} \to D^-_\infty(\mathcal{C})$ preserves cokernels of monomorphisms(e.g. $A \xrightarrow{\phi} B$) and $R^\bullet F\colon D^-_\infty(\mathcal{C}) \to D^-_\infty(\mathcal{C}')$ preserves cokernels.

$$\left(\begin{array}{c}\text{To be explicit, we can define } R^\bullet F \text{ either for } D_\infty \text{ or } D_1\text{, and these are related by } h\text{:}\\ \begin{array}{ccc} D^-_\infty(\mathcal{C}) & \xrightarrow{R^\bullet F} & D^-_\infty(\mathcal{C}') \\ \downarrow h & & \downarrow h \\ D^-_1(\mathcal{C}) & \xrightarrow{R^\bullet F} & D^-_1(\mathcal{C}') \end{array}\end{array}\right)$$

But cokernels(and colimits/limits in general) don't behave well in $D^-_1(\mathcal{C})$, and $i_0\colon \mathcal{C} \to D^-_1(\mathcal{C})$ does not preserve cokernels of monomorphisms in general(in particular of $A \xrightarrow{\phi} B$).
Workaround: Equip $D^-_1(\mathcal{C})$ with an extra structure to remember what morphisms were cokernels in $D^-_\infty(\mathcal{C})$. This is the structure of a **triangulated category**.

Rough Idea: A triangulated category is a (1-)category $\mathcal{C}$ equipped with a distinguished autoequivalence $[1]\colon \mathcal{C} \to \mathcal{C}$ and a class of distinguished compositions $X \xrightarrow{f} Y \to Z$ called **exact triangles**.

These would satisfy some axioms, in particular that Z is determined up to isomorphism by f, and that any morphism $f\colon X \to Y$ can be completed to an exact triangle.

We call Z a "mapping cone" of f and sometimes write it $\mathrm{cone}(f)$, but it does not satisfy any universal property.

We can define a triangulated structure on $D^-_1(\mathcal{C})$ by declaring mapping cones to be the images of cokernels in $D^-_\infty(\mathcal{C})$.

This gives a way of fixing our naive argument:

$$\begin{aligned} R^\bullet F(i_0 C) &\simeq R^\bullet F(i_0 \operatorname{cok}(A \xrightarrow{\phi} B)) && \text{cokernel in } \mathcal{C} \\ &\simeq R^\bullet F(\mathrm{cone}(i_0 A \xrightarrow{i_0\phi} i_0 B)) && \text{cokernel in } D^-_1(\mathcal{C}) \\ &\simeq \mathrm{cone}(R^\bullet F(i_0 A) \xrightarrow{R^\bullet F(i_0\phi)} R^\bullet F(i_0 B)). && \text{cokernel in } D^-_1(\mathcal{C}') \end{aligned}$$

This works: $i_0\colon \mathcal{C} \to D^-_1(\mathcal{C})$ take cokernels of monomorphisms to mapping cones and $R^\bullet F\colon D^-_1(\mathcal{C}) \to D^-_1(\mathcal{C}')$ preserves mapping cones(whenever we say a functor between triangulated categories is exact we also mean that this preserves mapping cones).

6. 9/10/2021

Next up: ∞-categories + simplicial set (ref: Lurie HTT Ch1)

- We said that topological categories capture the basic intuition of ∞-categories. but really we will formalize the latter differently using simplicial sets.
- Later we'll discuss a sense in which the two notions are equivalent.
- For $n \in \mathbb{N}$, we write $[n] := \{0, 1, \dots, n\}$

Definition 6.1. (1) The simplex category Δ has objects $\{[n]\}_{n\in\mathbb{N}}$ and morphisms (non-strictly) order-preserving maps.
(2) If $\mathcal{C}$ is a category, a simplicial object of $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. We write $\mathcal{C}_\Delta$ for the category of simplicial objects of $\mathcal{C}$.
(3) A cosimplicial object of $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$
(4) A simplicial set is a simplicial object of **Set**

- Explicitly, a simplicial set $S_\bullet$ is the data of
  - a set S_n for each $n \in \mathbb{N}$
  - a function $p^*\colon S_n \to S_m$ for each order-preserving function $p\colon [m] \to [n]$. Such that the p^* are compatible with composition.
- Some standard terminology/notation:
  - p^* is the "pullback along p"
  - for $j \in [n]$ the face map $d_j\colon S_n \to S_{n-1}$ is the pullback along

$$i \to \begin{cases} i & i \subset j \\ i+1 & i \supseteq j \end{cases}$$

$$\begin{tikzcd}
 & 1 \arrow[dl] & & 2 \arrow[dl] & \cdots & & j-1 \arrow[dl] & & j \arrow[dr] & & \cdots & n-2 \arrow[dr] & & n-1 \arrow[dr] & \\
1 & & 2 & \cdots & & j-1 & & j & & j+1 & \cdots & & n-1 & & n
\end{tikzcd}$$

  - for $j \in [n]$ the degeneracy map $f_j\colon S_n \to S_{n+1}$ is the pullback along

$$i \to \begin{cases} i & i \subseteq j \\ i-1 & i \supset j \end{cases}$$

$$\begin{tikzcd}
1 \arrow[dr] & & 2 \arrow[dr] & & \cdots & & j \arrow[dr] & & j+1 \arrow[dl] & & \cdots & & n \arrow[dl] & & n+1 \arrow[dl] \\
 & 1 & & 2 & & \cdots & & j & & \cdots & & n-1 & & n &
\end{tikzcd}$$

- Every order-preserving map is a composition of these, hence a simplicial set $S_\bullet$ is determined by the sets S_n together with its face and degeneracy maps.

-Idea behind the terminology

- Given $n \in \mathbb{N}$, we have the standard (topological) n-simplex

$$|\Delta^n| = \{(x_0, \dots, x_n) \in [0,1]^{n+1} \subset \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1\}$$

- for each $j \in [n]$ the inclusion of the jth face $|\Delta^{n-1}| \hookrightarrow |\Delta^n|$ is given by

$$(x_0, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{n-1})$$

- we also have the "degeneration" map $|\Delta^{n+1}| \to |\Delta^n|$ given by

$$(x_0, \dots, x_{n+1}) \mapsto (x_0, \dots, x_j + x_{j+1}, \dots, x_{n+1})$$

- one can define a similar map $|\Delta^m| \xrightarrow{p_*} |\Delta^n|$ for any order-preserving map $[m] \xrightarrow{p} [n]$ compatably with composition. Thus the $|\Delta^n|$ form a cosimplicial object of **Top**.

Key Example #1 :

Given a topological space X, we define the fundamental ∞-groupoid of X (or the singular complex of X) is the simplicial set defined $\Pi(X)_\bullet$ defined by:

- $\Pi(X)_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$
- given $p \colon [m] \to [n]$, we defined $p^* \colon \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X) \to \mathrm{Hom}_{\mathbf{Top}}(|\Delta^m|, X)$ by taking $f \colon |\Delta^n| \to X$ to $f \circ p_* \colon |\Delta^m| \to X$.

7. 9/13/21

Recall: Last time we introduced the simplex category by considering finite sets $[n]$ and order preserving maps.

Key example #0 The standard cosimplicial space $|\Delta^\bullet|$, i.e., $\Delta \xrightarrow{|\Delta^\bullet|} \mathbf{Top}$, $[n] \mapsto |\Delta^n|$

Key example #1. The fundamental ∞-groupoid $\Pi(X)_\bullet$ of a space X.

Quick definition: $\Pi(X)$ is defined as the composition

$$\Delta^{op} \xrightarrow{|\Delta^\bullet|^{op}} \mathbf{Top}^{op} \xrightarrow{\mathbf{Hom}_{\mathbf{Top}}(-,X)} \mathbf{Set}$$

Remark. Given a base point $x_0 \in X$, $\Pi(X)$ contains all the data needed to define $\pi_1(X, x_0)$.

• The point x_0 defines a 0-simplex $|\Delta^0| \xrightarrow{x_0} X$, where x_0 denotes the constant map in $\Pi(X)_0 = \mathbf{Hom}_{\mathbf{Top}}(|\Delta^0|, X)$.

• A path $\gamma : x_0 \to x_0$ defines a 1-simplex

$$|\Delta^1| \xrightarrow{\gamma} X$$

such that $d_0(\gamma) = d_1(\gamma) = x_0$.

• A homotopy h from γ_1 to γ_2 defines a 2-simplex

$$|\Delta^2| \xrightarrow{h} X$$

such that $d_1(h) = \gamma_1$, $d_2(h) = \gamma_2$, $d_3(h) = Id_{x_0}$.

Here is the picture of $|\Delta^2|$:

$$\begin{tikzcd} & \underset{\bullet}{1} \arrow[dr, "d_0(|\Delta_2|)"] & \\ \underset{0}{\bullet} \arrow[ur, "d_2(|\Delta_2|)"] \arrow[rr, "d_1(|\Delta_2|)"'] & & \underset{2}{\bullet} \end{tikzcd}$$

More generally, any 2-simplex $|\Delta^2| \xrightarrow{h} X$ defines a homotopy between $d_1(h)$ and $d_0(h) \circ d_2(h)$. Therefore, $\Pi(X)$ encodes the composition law for homotopy classes of paths.

Preview: If X is sufficiently nice, e.g., a C.W. complex, then we can recover the homotopy type of X from $\Pi(X)$.

Summary $\Pi(X)_\bullet$ extends $\pi_1(X, x_0)$ by allowing x_0 to vary and by remembering actual homotopies between paths rather than just the relation of homotopy equivalence.

Key example #2 Given a category $\mathcal{C}$, we define the **nerve** $\mathcal{N}(\mathcal{C}) \in \mathbf{Set}_\Delta$ as follows:

Let $\mathcal{N}(\mathcal{C})_n$ be the set of composable sequences of morphisms in $\mathcal{C}$.

$$\mathcal{N}(\mathcal{C})_n := \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n\}$$

.

The face map $d_j : \mathcal{N}(\mathcal{C})_n \to \mathcal{N}(\mathcal{C})_{n-1}$ is defined by the composition

$$d_j(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n) = A_0 \xrightarrow{f_1} \cdots \to A_{j-1} \xrightarrow{f_{j+1} \circ f_j} A_{j+1} \to \cdots \xrightarrow{f_n} A_n$$

8. 9/15/2021

Last time: more on the fundamental co-groupoid/singular complex, started discussing nerves.

Key example #2: Given a category $\mathcal{N}(\mathcal{C})$, we define $\mathcal{N}(\mathcal{C}) \in \mathbf{Set}_\Delta$ as follows:

-$\mathcal{N}(\mathcal{C})_n$ is the set of composable sequences of n morphims:

$$\mathcal{N}(\mathcal{C})_n = \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n\}$$

-The face map $d_j : \mathcal{N}(\mathcal{C})_n \to \mathcal{N}(\mathcal{C})_{n-1}$ is defined by composition:

$$d_j(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n) = A_0 \xrightarrow{f_1} A_1 \to \cdots \to A_j \xrightarrow{f_{j+1} \circ f_j} A_{j+1} \to \cdots \xrightarrow{f_n} A_n$$

-The degeneracy map $s_j : \mathcal{N}(\mathcal{C})_n \to \mathcal{N}(\mathcal{C})_{n+1}$ is defined using identity morphisms:

$$s_j(A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n) = A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_j} A_j \xrightarrow{id_{A_j}} A_j \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_n} A_n$$

-compatibility with composition in $\Delta \Leftrightarrow$ associativity and identity axioms for categories.

Quick definition: note that a poset (like $[n]$) is the same data as a category with at most one morphism between any two objects (i.e. $i \leq j \Leftrightarrow \mathrm{Hom}(i,j)$ nonempty), and order-preserving

maps are the same as functors.
$\Rightarrow$ the assignment $n \mapsto [n] \in Cat$ (This means category of category)defines a cosimplicial category $\Delta \xrightarrow{[\bullet]} Cat$.
-Then $\mathcal{N}(\mathcal{C})$ is just the composition

$$\Delta^{op} \xrightarrow{[\bullet]^{op}} \mathbf{Cat}^{op} \xrightarrow{\mathbf{Hom}_{\mathbf{Cat}}(-,\mathcal{C})} \mathbf{Set}$$

-The data of $\mathcal{C}$ is equivalent to the data of $\mathcal{N}(\mathcal{C})$:

$$Ob(\mathcal{C}) = \mathcal{N}(\mathcal{C})_0$$

$$Mor(\mathcal{C}) = \mathcal{N}(\mathcal{C})_1$$

$$composition \leftrightarrow d_1 : \mathcal{N}(\mathcal{C})_2 \to \mathcal{N}(\mathcal{C})_1$$

-The simplicial sets $\Pi(X)$, and $\mathcal{N}(\mathcal{C})$ suggest an analogy between spaces and categories:

$$points \leftrightarrow objects$$

$$paths \leftrightarrow morphisms$$

$$homotopies \quad between \quad paths \leftrightarrow ???$$

-Topological categories are one way of formalizing "homoptopies between morphisms" as "paths between points in a mapping space"
-There are other ways of formalizing the same idea.
-The one that we will focus on is that of **∞-category** or **quasicategories**.
-The idea: identify a class of simplicial sets which are close enough to being of the form $\mathcal{N}(\mathcal{C})$ to "do category theory with them" but which also include simplicial sets of the form $\Pi(X)$.
Q: Given $S. \in \mathbf{Set}_\Delta$, 1) how can we tell if $S. \cong \mathcal{N}(\mathcal{C})$ for some category $\mathcal{C}$? and 2) how can we tell if $S. \cong \Pi(X)$ for some space X?
Back to $\mathcal{N}(\mathcal{C})$: explicitly $\mathcal{N}(\mathcal{C})_2 = \{A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2\}$, $\mathcal{N}(\mathcal{C})_1 = \{A_0 \xrightarrow{f_1} A_1\}$, and $\mathcal{N}(\mathcal{C})_0 = \{A_0\}$.

$$d_0 : \mathcal{N}(\mathcal{C})_1 \to \mathcal{N}(\mathcal{C})_0, (A_0 \xrightarrow{f_1} A_1) \mapsto A_1$$

$$d_1 : \mathcal{N}(\mathcal{C})_1 \to \mathcal{N}(\mathcal{C})_0, (A_0 \xrightarrow{f_1} A_1) \mapsto A_0$$

$$d_0 : \mathcal{N}(\mathcal{C})_2 \to \mathcal{N}(\mathcal{C})_1, (A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2) \mapsto (A_1 \xrightarrow{f_2} A_2)$$

$$d_2 : \mathcal{N}(\mathcal{C})_2 \to \mathcal{N}(\mathcal{C})_1, (A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2) \mapsto (A_0 \xrightarrow{f_1} A_1)$$

Observation: the diagram

$$\begin{tikzcd}
\mathcal{N}(\mathcal{C})_2 \arrow[r, "d_2"] \arrow[d, "d_0"] & \mathcal{N}(\mathcal{C})_1 \arrow[d, "d_0"] \\
\mathcal{N}(\mathcal{C})_1 \arrow[r, "d_1"] & \mathcal{N}(\mathcal{C})_0
\end{tikzcd}$$

is Cartesian (i.e. $\mathcal{N}(\mathcal{C})_2 \simeq \mathcal{N}(\mathcal{C})_1 \times_{\mathcal{N}(\mathcal{C})_0} \mathcal{N}(\mathcal{C})_1$)
Note: for any $S. \in \mathbf{Set}_\Delta$ we have a commutative diagarm

$$\begin{tikzcd} S_2 \arrow[r, "d_2"] \arrow[d, "d_0"] & S_1 \arrow[d, "d_0"] \\ S_1 \arrow[r, "d_1"] & S_0 \end{tikzcd}$$

but it's usually not Cartesian.

9. 9/17/2021

Question: How do we characterize simplical sets of the form $N(\mathcal{C})$? Furthermore, how do we characterize of the form $\Pi(X)$?

To answer these two questions, we need to introduce the notion of a *horn.*

Definition 9.1. (1) For $n \in \mathbb{N}$, define

$$\Delta^n = \mathrm{Hom}_\Delta(-, [n]) \in \mathrm{Set}_\Delta := \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$$

(2) For $j \in [n]$, define $\Lambda^n_j \in \mathrm{Set}_\Delta$ by

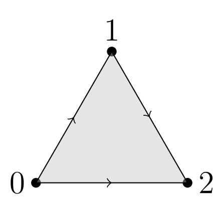
$$\begin{aligned} (\Lambda^n_j)_m &:= \{\text{order-perserving } p : [m] \to [n] \text{ such that } \{j\} \bigcup p([m]) \neq [n]\} \\ &\subset \{\text{order-perserving } p : [m] \to [n]\} \\ &= (\Delta^n)_m \end{aligned} \tag{9.2}$$

with face and degeneracy maps obtained by the restriction from Δ^n.

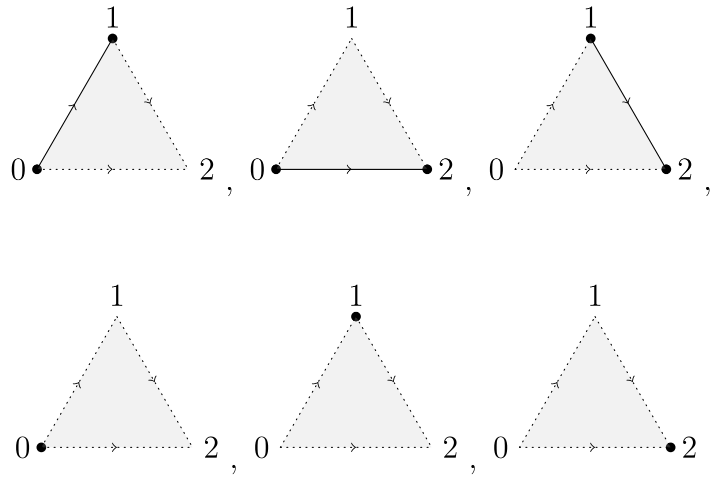
(3) In particular, we have a canonical monomorphism $\Lambda^n_j \hookrightarrow \Delta^n$ in Set_Δ. We call Λ^n_j the j^{th} *horn of* Δ^n.

(4) If $0 < j < n$, then we call Λ^n_j an *inner horn.*

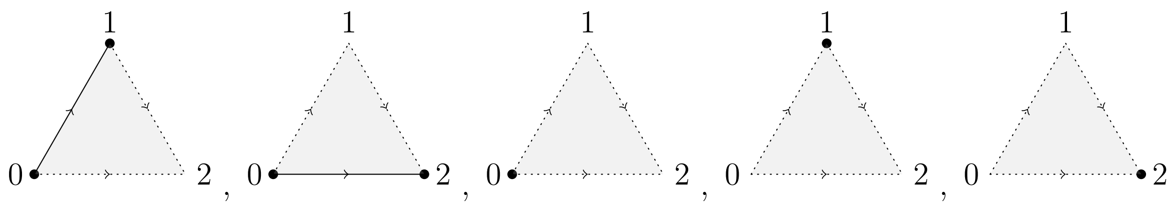
Remark. For any $K \in \mathrm{Set}_\Delta$, we have $K_n := \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, K)$ by the *Yoneda lemma.*
Example. Let's picture the elements of [2] as the vertices of $|\Delta^2|$:



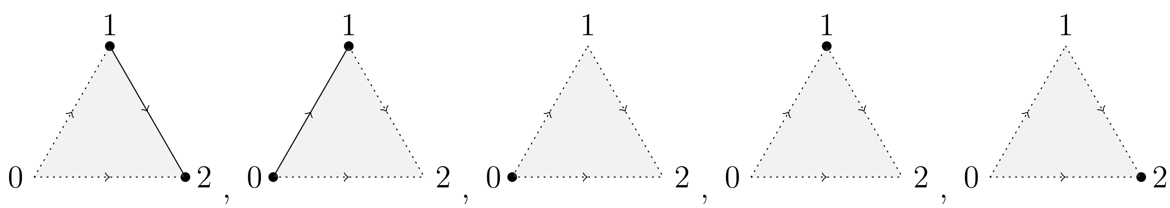
Then, we can picture the 1-simplices of Δ^2 and its horns. $(\Delta^2)_1 = \{p_2, p_1, p_0, q_0, q_1, q_2\}$ is a six-element set containing the following order-preserving maps:



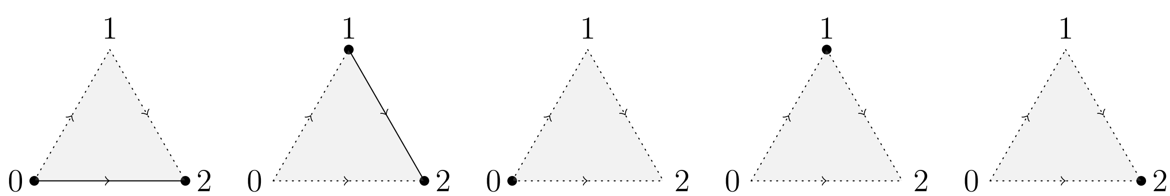
We call the three elements in the bottom row "degenerate simplices" because they are in the image of a degeneracy map. By definition of $(\Lambda^2_0)_1$, one can check that it is the following five maps:



Similarly, $(\Lambda^2_1)_1$ is:



Finally, $(\Lambda^2_2)_2$ is:



Proposition. (HTT 1.1.2.2) $K \in \mathrm{Set}_\Delta$ is of the form $N(\mathcal{C})$ for some category $\mathcal{C}$ if and only if whenever we have a morphism $\Lambda^n_j \to K$ with $0 < j < n$, there exists a *unique* morphism $\Delta^n \to K$ such that the following diagram commutes in Set_Δ:

$$\begin{tikzcd} \Lambda^n_j \arrow[r, hook] \arrow[dr] & \Delta^n \arrow[d, dashed, "\exists!"] \\ & K \end{tikzcd}$$

Example. Let's see why a morphism $\Lambda^2_1 \xrightarrow{\phi} N(\mathcal{C})$ satisfies this condition. On 1-simplices, ϕ is a function of the form:

$$p_2 \overset{\phi}{\mapsto} (A_0 \xrightarrow{f_1} A_1)$$

$$p_0 \overset{\phi}{\mapsto} (A_1 \xrightarrow{f_2} A_2)$$

$$q_0 \overset{\phi}{\mapsto} (A_0 \xrightarrow{id_{A_0}} A_0)$$

$$\vdots$$

The fact that ϕ is compatible with composition in Δ implies that the target of f_1 is equal to the source of f_2. Furthermore, it also means that degenerate simplices go to identity morphisms.

A morphism $\Delta^2 \xrightarrow{\overline{\phi}} N(\mathcal{C})$ is determined by $(\Delta^2)_2 \ni |\Delta^2| \mapsto (B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2) \in N(\mathcal{C})_2$. Since $\overline{\phi}$ is compatible with face maps, we have

$$p_2 \overset{\overline{\phi}}{\mapsto} (B_0 \xrightarrow{g_1} B_1)$$

$$p_0 \overset{\overline{\phi}}{\mapsto} (B_1 \xrightarrow{g_2} B_2)$$

$$p_1 \overset{\overline{\phi}}{\mapsto} (B_0 \xrightarrow{g_2 \circ g_1} B_2)$$

If $\overline{\phi}$ extends ϕ, this forces $A_0 = B_0$, $A_1 = B_1$, $A_2 = B_2$, $f_1 = g_1$, $f_2 = g_2$. So, $\overline{\phi}$ is unique if it exists.

10. 9/20/2021

We continue the discussion of characterizing nerves by horn-filling property. That is,

Proposition 10.1. *(HTT 1.1.2.2) Let $K \in Set_\Delta$ is of the form $N(\mathcal{C}) \Leftrightarrow$ Every j-th n-horn $\Lambda^n_j \to K$ with $0 < j < n$ extends uniquely to an n-simplex $\Delta^n \to K$*

Proof. (Idea) "Only If" direction: Define $Ob(\mathcal{C}) = K_0$, $Mor(\mathcal{C}) = K_1$, and source/target map $Mor(\mathcal{C}) \underset{s}{\overset{t}{\rightrightarrows}} Ob(\mathcal{C})$ as degeneracy maps $K_1 \underset{s}{\overset{t}{\rightrightarrows}} K_0$ and their composition from unique morphism extension property of the horn. That is, we have two different j-th n-horns and an unique extension to n-simplex defines a composition.

Q: We have witnessed characterization of nerve, which about the others? How to characterize when K is of the form $\Pi(X)$?

A: To address the question, it would be natural to construct a functor from topological space to simplicial sets. the following construction will be useful:

Proposition 10.2. *The functor $Top \xrightarrow{\Pi(\cdot)} Set$ denoted by $X \to \Pi(X)$ has a left adjoint $Set_\Delta \xrightarrow{|\cdot|} Top$ denoted by $K \to |K|$, we call $|K|$ the geometric realization of K*

Recall: this means for all $K \in Set_\Delta$ and $X \in Top$ we have a bijection between

$$Hom_{Set_\Delta}(K, \Pi(X)) \cong Hom_{Top}(|K|, X)$$

and these are compatible with composition.

Proof. (Idea) (Lurie, Goerss and Jardin, *Simplicial Homotopy Theory* Ch.1)
Define $\Delta \downarrow K$ a new category, the category of simplicies in K, by settings the objects to be $\sqcup_{n\in\mathbb{N}} K_n$ and $Hom_{\Delta\downarrow K}(\sigma,\tau) := \left\{ \begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[d, "\Theta"] & K \\ \Delta^m \arrow[ur, "\tau"] & \end{tikzcd} \right\}$ for all Θ, where $\sigma, \tau \in K_n, K_m$.

Remark 10.3. This set $Hom_{\Delta\downarrow K}(-,-)$ is a subset of morphism of $Hom_{Set_\Delta}(\Delta^n, \Delta^m) \cong Hom_{poset}([m],[n])$

Remark 10.4. This is really a general construction that takes any functor $F : \mathcal{C} \to Set$ to a category $C^{op} \downarrow F$

Note there is a forgetful functor $\Delta \downarrow K \to Set_\Delta$ that takes $\Delta^n \xrightarrow{\sigma} K$ to Δ^n

Now we want to take the colimit. The colimit of the diagram recover the original simplicial sets K, we give a general categorical fact (construction), in the spirit of Yoneda Lemma:

$$K \cong colim_{\Delta^n \xrightarrow{\sigma} K \text{ in } \Delta\downarrow K} \Delta^n$$

Now let's define the geometric realization by:

$$|K| \cong colim_{\Delta^n \xrightarrow{\sigma} K \text{ in } \Delta\downarrow K} |\Delta^n|$$

We can obtain what we want, the idea is to "glue" the simplicies together encoded in the structure in K, then we get some space.

For any space $X \in Top$ we have

$$\begin{aligned} Hom_{Top}(|K|, X) &\cong Hom_{Top}(colim_{\Delta^n \downarrow K} \Delta^n, X) \\ &\cong lim_{\Delta^n \downarrow K} Hom_{Top}(|\Delta^n|, X) \\ &\cong lim_{\Delta^n \downarrow K} Hom_{Set_\Delta}(|\Delta^n|, \Pi(X)) \\ &\cong Hom_{Set_\Delta}(colim_{\Delta^n \downarrow K} \Delta^n, \Pi(X)) \\ &\cong Hom_{Top}(|K|, \Pi(X)) \end{aligned}$$

Thus we are done.

11. 9/22/2021

Last time we talked about geometric realization. Roughly, given a simplicial set $K \in Set_\Delta$, we can get $|K| = \sqcup K_n \times |\Delta^n|/\text{gluing}$.

Proposition 11.1. $|-|$ *is left adjoint to* $\Pi(-)$.

- Note that $|\Delta^n|$ is the geometric realization of Δ^n, as $\Delta \downarrow \Delta^n$ has a final object $\Delta^n \xrightarrow{id} \Delta^n$. Then we have $|\Delta^n| = \text{colim}_{\Delta^k \to \Delta^n} |\Delta^k| = |\Delta^n|$.
- Similarly $|\Lambda^n_j|$ is isomorphic to the colimits over just its non-degenerate $(n-1)$-simplices and and their facets.

Example 11.2. $|\Lambda^2_0|$

- Note that $|\Lambda^2_0| \to |\Delta^2|$ is a retract: there is a continuous map $|\Delta^2| \to |\Lambda^2_0|$ s.t. $\Lambda^2_0| \to |\Delta^2| \to |\Lambda^2_0|$ is the identity.
- Fact: The same is true of $|\Lambda^n_j| \to |\Delta^n|$ for any $0 \leq j \leq n$.

Corollary 11.3. *For any space X, any morphism $|\Lambda^n_j| \to X (0 \leq j \leq n)$ extends to a morphism $|\Delta^n| \to X$.*

Corollary 11.4. *For any space X, any morphism $\Lambda^n_j \to \Pi(X)(0 \leq j \leq n)$ extends to a morphism $\Delta^n \to \Pi(X)$.*

The proof is by adjunction.

Definition 11.5. $K \in Set_\Delta$ is a Kan complex if it satisfies the extension condition in the above corollary.

- It is not true that every Kan complex is of the form $\Pi(X)$, but this is true up to homotopy.
- Recall that continuous maps $f, g : X \to Y$ are homotopic if there exists $h : X \times [0, 1] \to Y$

s.t. the diagram
$$\begin{tikzcd} X \times \{0\} \arrow[d, hook] \arrow[dr, "f"] & \\ X \times [0,1] \arrow[r, "h"] & Y \\ X \times \{1\} \arrow[u, hook] \arrow[ur, "g"'] & \end{tikzcd}$$
commutes.

Exercise: All limits and colimits exists in Set_Δ and are computed objectwise.

Definition 11.6. Two morphisms $f, g : J \to K$ in Set_Δ are homotopic if $\exists h : J \to \Delta^1 \to K$ such that

$$\begin{tikzcd} J \times \Delta^0 \arrow[d, hook, "id\times s_0"] \arrow[dr, "f"] & \\ J \times \Delta^1 \arrow[r, "h"] & K \\ J \times \Delta^0 \arrow[u, hook, "id\times s_1"'] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Fact: $\Pi(-)$ and $|-|$ take homotopic maps to homotopic maps.

12. 9/24/2021

Last time: We discussed Kan complex, homotopies.

Let $\mathbf{Kan} \subset \mathbf{Set}_\Delta$ be the full subcategory of Kan complexes and $H_0(\mathbf{Kan})$ the category with the same objects but

$$\mathrm{Hom}_{H_0(\mathbf{Kan})}(J, K) := \mathrm{Hom}_{\mathbf{Kan}}(J, K)/\sim, \text{ where } f \sim g \text{ if } f \text{ and } g \text{ are homotopic.}$$

Remark 12.1. The implicit proposition behind the definition is that simplicial homotopy is an equivalent relation and compatible with composition.

Similarly, let $CW \subset \mathbf{Set}$ be the full subcategory of $CW-$complexes, (for example, manifolds, $|K|$ for any $K \in \mathbf{Set}_\Delta$) and define $H_0(CW)$ similarly. (so for example any contractible space is isomorphic to a point in $H_0(CW)$.)

Theorem 12.2. *The adjoint functors $|-| : \boldsymbol{Set}_\Delta \to \boldsymbol{Top}$ and $\Pi(-) : \boldsymbol{Top} \to \boldsymbol{Set}_\Delta$ respect homotopy equivalence of morphisms and their restrictions induce inverse equivalences between $H_0(\boldsymbol{Kan})$ and $H_0(CW)$.*

Corollary 12.3. *For any $K \subset \boldsymbol{Kan}$, the canonical morphism $K \to \Pi(|K|)$ is invertible up to homotopy.*

Terminology: We often just call $H_0(CW) \simeq H_0(\mathbf{Kan})$ the homotopy category of spaces.

Now we state a key definition:

Definition 12.4. $K \in \mathbf{Set}_\Delta$ is an $\infty-$category if every morphism $\Lambda^n_j \to K$ with $0 < j < n$ extends to a morphism $\Delta^n \to K$.

Example 12.5. (1) $\mathcal{N}(\mathcal{C})$ is an ∞-category for any category $\mathcal{C}$.
(2) Any Kan complex (e.g. $\Pi(X)$) is an ∞-category.

If K is an ∞-category, we refer to elements of K_0 as objects and elements of K_1 as morphisms (or 1-morphisms). Given a morphism $F \in K_1$ we call $d_1(f)$ its source and $d_0(f)$ its target.

Informally: In an ordinary category we can say "$\phi : A \to C$ is **the**(unique) composition of $f : A \to B$ and $g : B \to C$", but in an $\infty-$category we can only say "$\phi : A \to C$ is **a** composition of $f : A \to B$ and $g : B \to C$", and by this we mean there exists $h \in K_2$ such that $d_2(h)$, $d_0(h) = g_1$ and $d_1(h) = \phi$. In pictures,

$$\begin{tikzcd} & B \arrow[dr, "g"] \arrow[d, phantom, "h"] & \\ A \arrow[ur, "f"] \arrow[rr, "\phi"'] & {} & C \end{tikzcd}$$

Now that we've defined ∞-categories, our next task is to extend the key notions of ordinary (and topological) category theory.

Opposites: If $\mathcal{C}$ is a category, $\mathcal{C}^{op}$ has the same objects but

$$\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

Definition 12.6. If $\mathcal{C}$ is an ∞-category, we define $\mathcal{C}^{op}$ by setting $\mathcal{C}^{op}_n := \mathcal{C}_n$ for all $n \in N$, and setting

$$(d_j : \mathcal{C}^{op}_n \to \mathcal{C}^{op}_{n-1} := d_{n-j} : \mathcal{C}_n \to \mathcal{C}_{n-1})$$

$$(s_j : \mathcal{C}^{op}_n \to \mathcal{C}^{op}_{n+1} := s_{n-j} : \mathcal{C}_n \to \mathcal{C}_{n+1})$$

13. 9/27/2021

Last time: We defined ∞-category.

Homotopy Categories (HTT 1.2.3)

Let $K \in \mathrm{Set}_\Delta$ and $f, g \in K$ be 1-simplices with the same faces i.e. $d_0(f) = d_0(g)$ and $d_1(f) = d_1(g)$.

Definition 13.1. f and g are homotopic if there exists $h \in K_2$ such that $d_2(h) = f$, $d_1(h) = g$, and $d_0(h) = s_0(d_0(f)) = Id_{d_0(f)}$ if K is an ∞-category.

$$\begin{tikzcd}
& d_0(f) \arrow[dr, "Id_{d_0(f)}"] \arrow[d, phantom, "h"] & \\
d_1(f) = d_1(g) \arrow[ur, "f"] \arrow[rr, "g"'] & & d_0(g) = d_0(f)
\end{tikzcd}$$

Theorem 13.2. *If K is an ∞-category, homotopy defines an equivalence relation on K_1. Moreover, there exists a unique category hK (the homotopy category of K) such that*

$$Ob(hK) := K_0 \textit{ and}$$
$$Mor(hK) := K_1/homotopy$$

and such that the natural functions

$$K_0 \simeq N(hK)_0 \textit{ and}$$
$$K_1 \longmapsto N(hK)_1 \textit{ i.e. the natural quotient map,}$$

extend to a morphism $K \longrightarrow N(hK)$ in Set_Δ (which is unique since the nerve of a category is determined by 0 and 1 simplices.)

Remark 13.3. The "functor" h(-): $Cat_\infty \longrightarrow Cat_1$ is a right adjoint of N(-): $Cat_1 \longrightarrow Cat_\infty$.

Informally, compositions in an ∞-category are not unique, but they are unique up to homotopy equivalence.

Definition 13.4. A morphism in an ∞-category K is an isomorphism (or equivalence) if its image in hK is an isomorphism in the usual sense.

Equivalently a morphism $f : X \longrightarrow Y$ in K is an isomorphism if there exists a morphism $f^{-1} : Y \longrightarrow X$ and a 2-simplex $h \in K_2$ such that

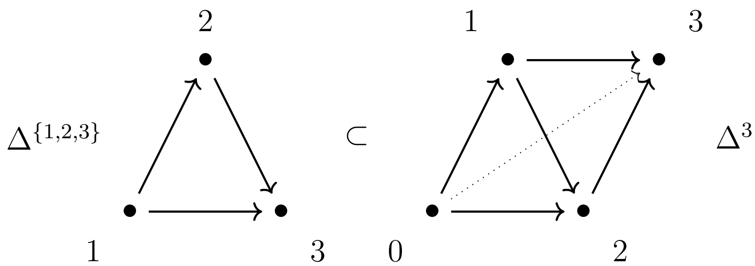
$$\begin{tikzcd}
& Y \arrow[dr, "f^{-1}"] \arrow[d, phantom, "h"] & \\
X \arrow[ur, "f"] \arrow[rr, "Id_X := S_0(X)"'] & & X
\end{tikzcd}$$

Proposition 13.5. *An ∞-category K is a Kan complex if and only if every morphism in K is an isomorphism.*

Mapping Spaces (HTT 1.2.2)

Part of the analogy comparing ∞-categories to topological categories is that we can still define a mapping space between objects in an ∞-category, but it's only canonical up to homotopy equivalence.

Notation: given a subset of the interval $I \subset [n]$, let $\Delta^I \subset \Delta^n$ be the largest simplicial subset with $\Delta^I_0 = I$.



14. 9/29/2021

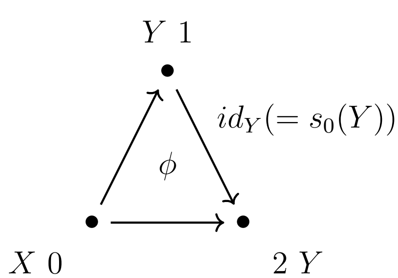
Last time: homotopy category
Next up: mapping space
Notation: given $I \subset [n]$, let $\Delta^I \subset \Delta^n$ be the largest simplicial subset with $\Delta^I_0 = I \subset [n] = \Delta^n_0$ (abstractly $\Delta^I \simeq \Delta^{|I|}$).

Definition 14.1. Let $K \subset Set_\Delta$ and $X, Y \in K_0$. Define $Hom^L_K \in Set_\Delta$ by

$$Hom^L_K(X,Y)_n = \text{maps } \phi : \Delta^{n+1} \longrightarrow K \text{ such that the following diagrams commute:}$$

$$\begin{array}{ccc} \Delta^{\{0\}} & \xrightarrow{surj} & \Delta^0 \\ \big\downarrow{\scriptstyle incl} & & \big\downarrow{\scriptstyle incl\ X} \\ \Delta^{n+1} & \xrightarrow{\phi} & K \end{array} \qquad \begin{array}{ccc} \Delta^{\{1,\ldots,n+1\}} & \xrightarrow{surj} & \Delta^0 \\ \big\downarrow{\scriptstyle incl} & & \big\downarrow{\scriptstyle incl\ Y} \\ \Delta^{n+1} & \xrightarrow{\phi} & K \end{array}$$

Example 14.2. An element of $Hom^L_K(X,Y)_1$ is a 2-simplex $h : \Delta^2 \longrightarrow K$ of the following form



i.e. ϕ is a homotopy between a pair of 1-simplicies ($d_2(\phi)$ and $d_1(\phi)$) with vertices X and Y.

Exercise. Define the face/degeneracy maps of $Hom^L_K(X,Y)$.

Proposition 14.3. *If K is an ∞-category, then $Hom^L_K(X,Y)$ is a Kan complex. We call it the space of left morphisms from X to Y, and its image in $H_0(Kan) \simeq H_0(CW)$ the mapping space from X to Y.*

Remark 14.4. The definition of $Hom_K^L(X,Y)$ was not canonical (i.e. it involved arbitrary choices), but it turns out to be canonical up to homotopy.

<u>**Functors**</u> (HTT Sec. 1.2.7)

Exercise. If C and D are ordinary categories, there is a canonical bijections between functors $C \longrightarrow D$ and morphisms $\mathcal{N}(C) \longrightarrow \mathcal{N}(D)$ in Set_Δ.

Definition 14.5. If J and K are ∞-categories, we will call a morphism $J \longrightarrow K$ in Set_Δ a functor from J to K.

- In ordinary category theory, we define a category $Fun(C,D)$ whose objects are functors and whose morphisms are natural transformations.
- We generalize this construction to ∞-categories as follows:

Definition 14.6. Given $J, K \in Set_\Delta$, let $Map_{Set_\Delta}(J,K) \in Set_\Delta$ be the functor $\Delta^{op} \longrightarrow Set$ that takes $[n]$ to $Hom_{Set_\Delta}(J \times \Delta^n, K)$, with structure maps defined by the Yoneda embedding $[n] \longrightarrow \Delta^n$.

Proposition 14.7. *If J and K are ∞-categories, so is $Map_{Set_\Delta}(J,K)$. In this case we also write it as $Fun(J,K)$ and call it the ∞-category of functors from J to K.*

Exercise. $Fun(J,K)_0 := Hom_{Set_\Delta}(J \times \Delta^0, K)$. But $J \times \Delta^0 \simeq J$ for any J, so this is just $Hom_{Set_\Delta}(J,K)$.

15. 10/1/2021

Last time: Mapping spaces, functors

Definition 15.1. (Proposition) If J and K are ∞-categories, the ∞-category $\mathrm{Fun}(J,K)$ of functors from J to K is defined by

$$\mathrm{Fun}(J,K)_n := \mathrm{Hom}_{\mathrm{Set}_\Delta}(J \times \Delta^n, K).$$

Example 15.2. What is a 1-simplex $\phi \in \mathrm{Fun}(J,K)$, explicitly?

It is a morphism $J \times \Delta^1 \to K$ in Set_Δ. It defines functors $F, G \colon J \to K$ via

$$\begin{tikzcd}
J \simeq J \times \Delta^0 \arrow[d, "id_J \times d_1"] \arrow[dr, "F"] & \\
J \times \Delta^1 \arrow[r, "\phi"] & K \\
J \simeq J \times \Delta^0 \arrow[u, "id_J \times d_0"'] \arrow[ur, "G"'] &
\end{tikzcd}$$

Note that for any n we have (i) $(J \times \Delta^1)_n \cong J_n \times \Delta^1_n$ and (ii) $\phi_n \colon (J \times \Delta^1)_n \to K_n$. Recall that $K_1 = \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, K) \Rightarrow (\Delta^1)_1 = \mathrm{Hom}(\Delta^1, \Delta^1)$, which contains id_{Δ^1}. For each $X \in J_0$, ϕ also defines a morphism $F(X) \xrightarrow{\phi_X} G(X)$ in K by the formula $\phi_X := \phi_1(s_0(X), id_{\Delta^1}) \in K_1$, where $id_X := s_0(X)$. We see that $F(x), G(x) \in K_0$, $\phi_X \in K_1$, and $d_1(\phi_X) = F(X), d_0(\phi_X) = G(X)$.

Exercise. when $J \simeq N(\mathcal{C})$ and $K \simeq N(\mathcal{D})$, the morphisms ϕ_X define a natural transformation from F to G. This construction defines a bijection between $\mathrm{Fun}(N(\mathcal{C}), N(\mathcal{C}))_1$ and natural transformations of functors from $\mathcal{C}$ to $\mathcal{D}$.

15.1. **Initial and Final Objects.**

Definition 15.3. An object X in a category $\mathcal{C}$ is **initial** (resp. **final**) if for every object Y, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (resp. $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ has a single element.

- I.e. "$\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a point"
- To extend this to the setting of ∞-categories, we just reinterpret it as a statement about mapping spaces.

Definition 15.4. An object X of an ∞-category K is **initial** (resp. **final**) if for every object Y, the mapping space $\mathrm{Map}_K(X, Y)$ (resp. $\mathrm{Map}_K(Y, X)$) is contractible.

Limits and Colimits

- Let $p \colon I \to \mathcal{C}$ be a functor between ordinary categories.

-The **overcategory of** p is the category $\mathcal{C}_{/p}$ with objects $\mathrm{Ob}(\mathcal{C}/p) = \{(x, \{f_i \colon X \to x_i := p(i)\}_{i \in I})$ such that

$$\begin{tikzcd} X \arrow[r] & X_j \\ & X_i \arrow[u, "p(g)"'] \end{tikzcd}$$

commutes for every $g \in \mathrm{Hom}_I(i, j)$ and all $i, j \in I\}$, for $x \in$ $\mathrm{Ob}(\mathcal{C})$ and $f_i \in \mathrm{Mor}(\mathcal{C})$. And $\mathcal{C}_{/p}$ has morphisms $\mathrm{Hom}_{\mathcal{C}_{/p}}((x, \{f_i\}), (x', \{f'_i\})) = \{\phi \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "r_i"'] & & X' \arrow[dl, "r'_i"] \\ & X_i & \end{tikzcd}$$

commutes for all $i \in I\}$.

Definition 15.5. A **limit of** p is a final object of the overcategory $\mathcal{C}_{/p}$.

Remark 15.6. if $(X, \{f_i\})$ is a limit of p_i we often just say "X is a limit of p" and leave the $\{f_i\}$ implicit.

16. 10/4/2021

Last time: ∞-category Fun(J,K), initial and final objects, limits and colimits

Now we need to figure out suitable definitions for initial and final objects as well as limits and colimits for ∞-categories. We cannot directly duplicate these definition from ordinary categories because the definition of morphisms in the overcategory requires us to refer to the composition of two morphisms and check that it agrees with another. However, in an ∞-category the composition of two morphisms is not defined uniquely but only up to homotopy.

The limit of an ∞-category $\mathcal{C}$ as an ∞-category is generally more interesting than the ordinary limit of the its homotopy category $h(\mathcal{C})$. The limit will incorporate information from all the simplices of $\mathcal{C}$ rather than just the 0-simplices and 1-simplices used in the limit of $h\mathcal{C}$. We give an explicit example of a computation of a limit in the usual category sense using the overcategory $\mathcal{C}/p$ to help motivate the definition we will see next time for ∞-categories.

We will use the following example throughout the next few lectures. Let the category I be

$$I = \begin{tikzcd} & \bullet\, b \arrow[d, "bc"] \\ a\,\bullet \arrow[r, "ac"'] & \bullet\, c \end{tikzcd}.$$

Explicitly the objects of I are $\mathrm{Ob}(I) = \{a,b,c\}$ and the morphisms of I are $\mathrm{Mor}(I) = \{ac, bc, aa, bb, cc\}$. Let $p : I \to \mathcal{C}$. Then by definition the set of objects of the overcategory $\mathcal{C}/p$ is

$$\mathrm{Ob}(\mathcal{C}_{/p}) = \left\{ (X, f_a, f_b) \text{ such that } \begin{tikzcd} X \arrow[r, "f_b"] \arrow[d, "f_a"'] & p(b) \arrow[d, "p(bc)"] \\ p(a) \arrow[r, "p(ac)"'] & p(c) \end{tikzcd} \text{ commutes} \right\}.$$

Note an object of $\mathcal{C}/p$ also has a map $f_c : X \to p(c)$ but f_c is determined by f_a and f_b, so we omit it to simplify the diagrams. We now also can explicitly write the morphisms in $\mathcal{C}_p$ as

$$\mathrm{Mor}(\mathcal{C}/p) = \left\{ (X', f'_a, f'_b, \phi) \text{ such that } \begin{tikzcd} X' \arrow[dr, "\phi"] \arrow[drr, bend left, "f'_b"] \arrow[ddr, bend right, "f'_a"'] & & \\ & X \arrow[r, "f_b"] \arrow[d, "f_a"] & p(b) \arrow[d, "p(bc)"] \\ & p(a) \arrow[r, "p(ac)"] & p(c) \end{tikzcd} \text{ commutes} \right\}.$$

Then by definition a final object of $\mathcal{C}_{/p}$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f_b"] \arrow[d, "f_a"'] & X_b \arrow[d] \\ X_a \arrow[r] & X_c \end{tikzcd}$$

such that given any other commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'_b"] \arrow[d, "f'_a"'] & X'_b \arrow[d] \\ X'_a \arrow[r] & X'_c \end{tikzcd}$$

there exists a unique morphism $\phi \in \mathrm{Mor}(\mathcal{C}_{/p})$ with $\phi : X' \to X$. This means that the diagram

$$\begin{tikzcd} X' \arrow[dr, "\phi"] \arrow[drr, bend left, "f'_b"] \arrow[ddr, bend right, "f'_a"'] & & \\ & X \arrow[r, "f_b"] \arrow[d, "f_a"] & X_b \arrow[d] \\ & X_a \arrow[r] & X_c \end{tikzcd}$$

commutes. Then the final object in $\mathcal{C}_{/p}$ is exactly the usual pullback along the maps $p(ac)$ and $p(bc)$. Thus we have realized the pullback as a limit over a diagram of shape I.

We cannot just replace "category" by "∞-category" in the definition of the overcategory. The main problem is that composition is not defined uniquely in ∞-categories. Instead, we should replace the condition that $p(g)' \circ f_i = f_j$ in the condition on objects in the overcategory with the condition that there exists a 2-simplex whose boundary one direction is $p(g) \circ f_i$ and the boundary the other direction is f_j for every $g : i \to j$. As a diagram this is

$$\begin{tikzcd} & p(i) \arrow[dr, "p(g)"] & \\ X \arrow[ur, "f_i"] \arrow[rr, "f_j"'] & & p(j) \end{tikzcd}$$

where the shaded region in the middle is the 2-simplex whose existence is a condition for an object (X, f_i, f_j) to be in the overcategory. This idea will be made more rigorous in the next few sections using the idea of a join.

17. 10/6/2021

Last time we defined limits and colimits in ordinary categories. In order to define limits and colimits in ∞-categories, like before, we want to first wirte down a suitable definition of overcategories; that requires the notion of a ***join***. By convention, for all $K \in \mathrm{Set}_\Delta$, define $K_{-1} = \mathrm{pt}$.

Definition 17.1. Given $J, K \in \mathrm{Set}_\Delta$, the join $J * K \in \mathrm{Set}_\Delta$ is defined by setting

$$\begin{aligned}(J * K)_n = \coprod_{m=-1}^{n} J_m \times K_{n-m-1} &= \coprod_{\substack{m+l=n-1 \\ -1 \le m, l \le n}} J_m \times K_l \\ &= (J_{-1} \times K_n) \amalg (J_0 \times K_{n-1}) \amalg \cdots \amalg (J_n \times K_{-1}) \\ &= K_n \amalg (J_0 \times K_{n-1}) \amalg \cdots \amalg J_n,\end{aligned}$$

and for $(j, k) \in J_m \times K_{n-m-1}$ setting the degenercy and face maps to be

$$d_i(j,k) = \begin{cases} \big(d_i(j), k\big) \in J_{m-1} \times K_{n-m-1} & i \le m \\ \big(j, d_{i-m-1}(k)\big) \in J_m \times K_{n-m-2} & i \ge m+1 \end{cases}$$

Similarly for $s_i(j,k)$.

Note that J and K are naturally simplicial subsets of $J * K$.

Definition 17.2. Given categories $\mathcal{C}$ and $\mathcal{D}$, the join $\mathcal{C} * \mathcal{D}$ is the category defined by

$$\mathrm{Ob}(\mathcal{C} * \mathcal{D}) := \mathcal{C} \amalg \mathcal{D}$$
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(X,Y) := \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X,Y) & X, Y \in \mathcal{C}; \\ \mathrm{Hom}_{\mathcal{D}}(X,Y) & X, Y \in \mathcal{D}; \\ \mathrm{pt} & X \in \mathcal{C},\ Y \in \mathcal{D}; \\ \varnothing & X \in \mathcal{D},\ Y \in \mathcal{C}. \end{cases}$$

Then $\mathcal{N}(\mathcal{C} * \mathcal{D}) = \mathcal{N}(\mathcal{C}) * \mathcal{N}(\mathcal{C})$.

Proposition 17.3. *If J and K are ∞-categoriues, so is $J * K$.*

Remark 17.4. We can rephrase our definition of overcategory $\mathcal{C}_{/p}$ in terms of joins:

- Regard $[n]$ as a certain category with at most one morphism between any two objects,
- The objects of $\mathcal{C}_{/p}$ are functors $[0] * I \to \mathcal{C}$ whose restriction to I coincides with p.
- For example, if

$$I = \begin{tikzcd} & \bullet\, b \arrow[d, "bc"] \\ a\, \bullet \arrow[r, "ac"'] & \bullet\, c \end{tikzcd} \qquad [0] * I = \begin{tikzcd} 0\, \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet\, b \arrow[d, "bc"] \\ a\, \bullet \arrow[r, "ac"'] & \bullet\, c \end{tikzcd}$$

  Then a diargam in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & p(b) \arrow[d, "p(bc)"] \\ p(a) \arrow[r, "p(ac)"'] & p(c) \end{tikzcd}$$

  is really a picture of a functor $[0] * I \to \mathcal{C}$ whose restriction to I is given by p.
- Similarly, a morphism in $\mathcal{C}_{/p}$ is a functor $[1] * I \to \mathcal{C}$ whose restriction to I coincides with p. In above example,

$$[1] * I = \begin{tikzcd} \overset{0}{\bullet} \arrow[dr, "01"] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, bend right] & & \\ & \overset{1}{\bullet} \arrow[r] \arrow[d] \arrow[dr] & \bullet\, b \arrow[d, "bc"] \\ & \underset{a}{\bullet} \arrow[r, "ac"] & \underset{c}{\bullet} \end{tikzcd}$$

So a diagram

$$\begin{tikzcd}
X' \arrow[dr, "\phi"] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, bend right] & & \\
& X \arrow[r] \arrow[d] \arrow[dr] & p(b) \arrow[d, "p(bc)"] \\
& p(a) \arrow[r, "p(ac)"'] & p(c)
\end{tikzcd}$$

is a picture of a functor $[1] * I \to \mathcal{C}$.

Definition 17.5. Let $p : J \to K$ be a morphism in Set_Δ. Then $K_{/p} \in \mathrm{Set}_\Delta$ is defined by

$$\left(K_{/p}\right)_n = \left\{\phi \in \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n * J,\ K) \text{ such that } \phi|_J \equiv p\right\}$$

with structure maps defined by the Yoneda embedding $\Delta^{\mathrm{op}} \to \mathrm{Set}_\Delta$, $[n] \mapsto \Delta^n$.

Exercise.
Let $\mathcal{N}(p) : \mathcal{N}(I) \to \mathcal{N}(\mathcal{C})$ be the morphism associated to a functor $p : I \to \mathcal{C}$. Then

$$\mathcal{N}(\mathcal{C}_{/p}) \cong \mathcal{N}(K)_{/\mathcal{N}(p)}.$$

Proposition 17.6. *If K is an ∞-category, then so is $K_{/p}$.*

Definition 17.7. If $p : J \to K$ is a morphism in Set_Δ and K is an ∞-category, we call a final object of $K_{/p}$ a ***limit of p***.

18. 10/13/2021

Last time: extended example

-Given a space X and a point $x_0 \in X$, we can construct the limit of

$$\begin{tikzcd}
& pt \arrow[d, "x_0"] \\
pt \arrow[r, "x_0"'] & X
\end{tikzcd} .$$

in $Ho(\mathbf{Top})$ or $\mathbf{Top}_\infty$.

-Recall that a 2-simplex $\mathbf{Top}_\infty$ is pictured as

$$\begin{tikzcd}
& \bullet\, Y \arrow[dr, "f_{YZ}"] & \\
X\, \bullet \arrow[ur, "f_{XY}"] \arrow[rr, "f_{XZ}"'] & \arrow[u, phantom, "h"] & \bullet\, Z
\end{tikzcd}$$

where X, Y, Z are spaces, f_{XY}, f_{YZ}, f_{XZ} are continuous maps and $h : X \times [0,1] \to Z$ such that $h|_{X\times\{0\}} = f_{YZ} \circ f_{XY}$ and $h|_{X\times\{1\}} = f_{XZ}$.

-Let $p : \mathcal{N}(I) \to \mathbf{Top}_\infty$ be as above. We have

$Ob\big(\mathbf{Top}_{\infty/p}\big) = \Big\{\phi \in \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{N}([0] * I) \simeq \Delta^0 * \mathcal{N}(I),\ \mathbf{Top}_\infty) \text{ such that } \phi|_{\mathcal{N}(I)} \equiv p.\Big\}$

-Let's unpack $\mathcal{N}([0] * I)$: $I =$

$$\begin{tikzcd} & b \arrow[d, "bc"] \\ a \arrow[r, "ac"'] & c \end{tikzcd} \quad . \quad [0] = \cdot^0$$

$$(\Delta^0 * \mathcal{N}(I))_0 = \{0, a, b, c\}$$

$$\begin{aligned}(\Delta^0 * \mathcal{N}(I))_1 =& \Delta^0_1 \cup (\Delta^0_0 \times \mathcal{N}(I)_0) \cup \mathcal{N}(I)_1 \\ =& \{0a, 0b, 0c, ac, bc\} + \text{degenerate 1-simplices}\end{aligned}$$

$$\begin{aligned}(\Delta^0 * \mathcal{N}(I))_2 =& \Delta^0_2 \cup (\Delta^0_1 \times \mathcal{N}(I)_0) \cup (\Delta^0_0 \times \mathcal{N}(I)_1) \cup \mathcal{N}(I)_2 \\ =& \{0ac, 0bc\} + \text{degenerate 1-simplices}\end{aligned}$$

-Thus we can picture $\Delta^0 * \mathcal{N}(I)$ as

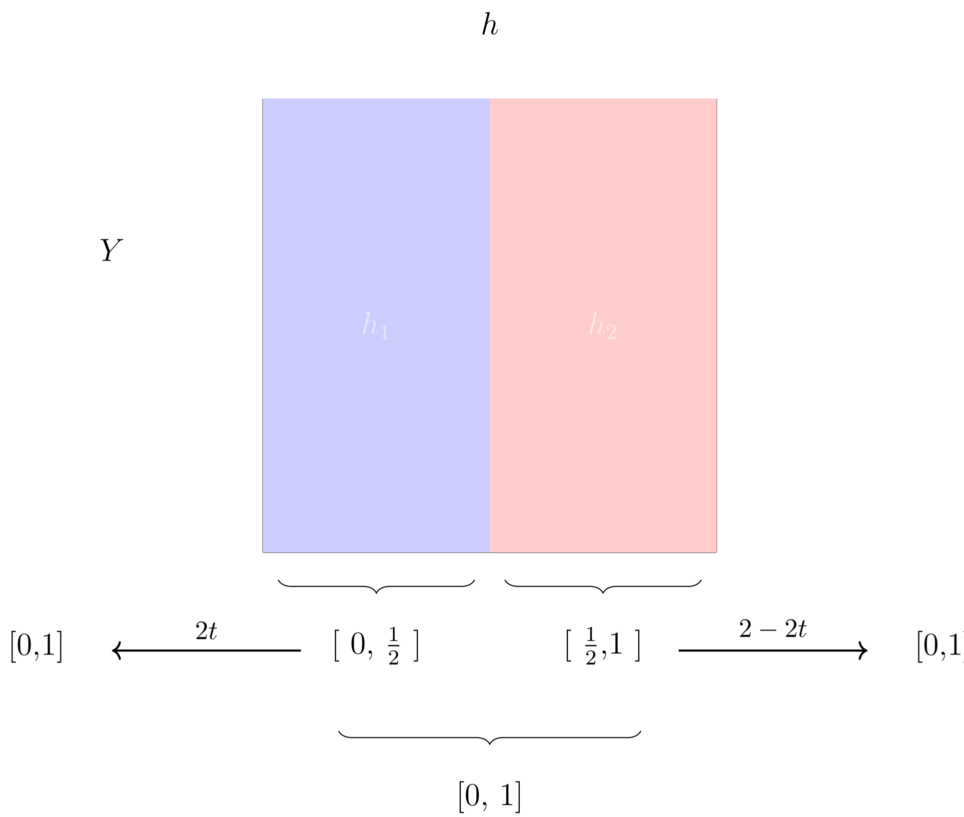
$$\begin{tikzcd} 0 \arrow[r, "0b"] \arrow[d, "0a"'] \arrow[dr, "0c" description] \arrow[d, phantom, shift left=4ex, "0ac"] & b \arrow[d, "bc"] \arrow[dl, phantom, shift right=3ex, "0bc"] \\ a \arrow[r, "ac"'] & c \end{tikzcd}$$

-Thus let's see that

$$Ob(\mathbf{Top}_{\infty/p}) = \{\text{diagrams} \quad \begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, "f" description] \arrow[d, phantom, shift left=3ex, "h_2"] & pt \arrow[d, "x_0"] \arrow[dl, phantom, shift right=3ex, "h_1"] \\ pt \arrow[r, "x_0"'] & X \end{tikzcd} \quad \text{in } \mathbf{Top}_\infty\}$$

$$\begin{aligned}= \{(Y, f, h_1, h_2) &\text{ where } f : Y \to X \text{is a continuous} \\ &\text{map and } h_1, h_2 : Y \times [0,1] \to X \text{ such that} \\ &h_1|_{Y\times\{0\}} = h_2|_{Y\times\{0\}} \equiv x_0 \text{ and } h_1|_{Y\times\{1\}} = h_2|_{Y\times\{1\}} = f\}\end{aligned}$$

-The natural functor functor $\mathbf{Top}_{\infty/p} \to Ho(\mathbf{Top})_{/p}$ forgets the data of h_1 and h_2, and just remember the property that f was nullhomotopic.
-We can "glue h_1 and h_2 together" to get a continuous map $h : Y \times [0,1] \to X$ such taht $h|_{Y\times\{0\}} = h|_{Y\times\{1\}} \equiv x_0$. Here 's the picture:



-Thus loses no information, so

$$Ob(\mathbf{Top}_{\infty/p}) = \{(Y,h) \text{ where } h : Y \times [0,1] \to X \text{ is a map}$$
$$\text{such that } h|_{Y\times\{0\}} = h|_{Y\times\{1\}} \equiv x_0\}$$

-Aside: gives spaces X, Y, the space $Maps(X,Y)$ is characterized by there being (functorial) bijection

$$Hom_{Top}(Z, Maps(X,Y)) \simeq Hom_{Top}(Z \times X, Y)$$

for all spaces Z.

-Thus we also have

$$Ob\big(\mathbf{Top}_{\infty/p}\big) = \left\{(Y, \bar{h}) \text{ where } \bar{h} : Y \to \Omega_{x_0} X\right\}$$

where the based loop space $\Omega_{x_0}X$ is the subspace of $Maps([0,1],X)$ consisting of paths beginning and ending at x_0.

(Ex: the fundamental group $\pi_1(X,x_0)$ is the set of connected components of $\Omega_{x_0}X$)

-If we follow the definition of morphisms $\mathbf{Top}_{\infty/p}$ through this, we find that

$$Mor(\mathbf{Top}_{\infty/p}) = \left\{ \text{commutative diagrams } \begin{tikzcd} Y' \arrow[d, "\phi"'] \arrow[dr, "\bar{h}'"] & \\ Y \arrow[r, "\bar{h}"'] & \Omega_{x_0}X \end{tikzcd} \right\}$$

-Thus the final object of $\mathbf{Top}_{\infty/p}$ is $\Omega_{x_0}X \xrightarrow[id]{} \Omega_{x_0}X$, because any map $Y \xrightarrow[\bar{h}]{} \Omega_{x_0}X$ extends uniquely to a commutative diagram

$$\begin{tikzcd} Y \arrow[d, "\bar{h}"'] \arrow[dr, "\bar{h}"] & \\ \Omega_{x_0}X \arrow[r, "id"'] & \Omega_{x_0}X \end{tikzcd}$$

$\Rightarrow$ the limit of p in $\mathbf{Top}_\infty$ is $\Omega_{x_0}X$, which is much more interesting than just a point!

19. 10/18/2021

Last time we talked about limits and loop spaces.

Definition 19.1. Given a morphism $p: J \to K$ in Set_Δ. The **undercategory** $K_{p/}$ of p is defined by $(K_{p/})_n = \{\phi \in Hom_{Set_\Delta}(J * \Delta^n, K) \text{ such that } \phi|_J = p\}$

A **colimit** of p is an initial object of $K_{p/}$. Or sometimes, it is just the image of $O \in \Delta^0_0 \subset (J * \Delta^0)_0$ in K_0 under an initial object $\phi: J * \Delta^0 \to K$.

Exercise:

$$\Delta^0 * N\left(\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}\right) \simeq \Delta^1 \times \Delta^1 \simeq N\left(\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}\right) * \Delta^0$$

Terminology A diagram $\Delta^1 \times \Delta^1 \to K$ in an ∞-category K is **Cartesian** or a **pullback**

square if it is the limit of its restriction to $N\left(\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}\right)$ and **coCartesian** or a **pushout square** if it is the colimit of its restriction to $N\left(\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}\right)$

Example 19.2. Given a space X, let's compute the colimit of $\begin{tikzcd} X \arrow[r] \arrow[d] & pt \\ pt & \end{tikzcd}$ in $\mathbf{Top}_\infty$

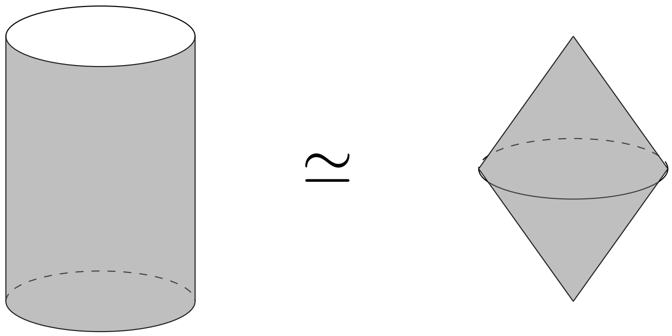
Following the analysis of last time, we have

$$\begin{aligned} Ob(\mathbf{Top}_\infty)_{p/} &= \left\{\text{diagrams } \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "f"] \arrow[dr, phantom, shift left=3, "h_2"] \arrow[dr, phantom, shift right=3, "h_1"'] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & Y \end{tikzcd} \text{ in } \mathbf{Top}_\infty\right\} \\ &= \{(Y, f, h_1, h_2) \text{ such that } f : X \to Y \text{and } h_1, h_2 : X \times [0,1] \to Y \\ &\quad \text{are continuous maps with } h_1|_{X\times\{0\}} = h_2|_{X\times\{0\}} = f \\ &\quad \text{and } h_1|_{X\times\{1\}}, h_2|_{X\times\{1\}} \text{ are constant maps } \} \\ &= \{(Y, h) \text{ such that } h : X \times [0,1] \to Y \text{ is continuous and such that} \\ &\quad h|_{X\times\{0\}} \text{ and } h|_{X\times\{1\}} \text{ are constant}\} \end{aligned}$$

$$Mor(Top_\infty)_{p/} = \left\{ \text{commutative diagrams } \begin{tikzcd} & Y \arrow[d] \\ X \times [0,1] \arrow[ur] \arrow[r] & Y' \end{tikzcd} \right\}$$

This implies that the initial object of $(\mathbf{Top}_\infty)_{p/}$ is the projection $X \times [0,1] \to X \times [0,1]/(p,0)\ (q,0), (p,1)\ (q,1)$ for all $p, q \in X$. This is called the **suspension** of X and is denoted ΣX.

Exercise $\Sigma S^n \simeq S^{n+1}$ for any $n \in \mathbb{N}$.



A similar computation to last time shows that the colimit of $\begin{tikzcd} X \arrow[r] \arrow[d] & pt \\ pt & \end{tikzcd}$ in **Ho(Top)** is again just a point.

Fact: $X \simeq \Omega_{x_0}\Sigma X$ for any space X. This is a weak homotopy equivalence.

20. 10/22/2021

Last time: Pointed spaces and zero objects
Today Digression: The ∞-category of ∞-categories (and of spaces)

Definition 20.1. A **simplicial category** is a category enriched in simplicial sets. I.e. it is the data of

- a collection $Ob(\mathcal{C})$ of objects
- for each $X, Y \in Ob(\mathcal{C})$ a simplicial set $Map_{\mathcal{C}}(X,Y) \in Set_\Delta$
- for each $X, Y \in Ob(\mathcal{C})$ a morphism
$$Map_{\mathcal{C}}(X,Y) \times Map_{\mathcal{C}}(Y,Z) \to Map_{\mathcal{C}}(X,Z)$$
in Set_Δ, which are collectively associative.

We write Cat_Δ for the category of simplicial categories and simplicial functors.

Example 20.2. Given any $J, K \in Set_\Delta$, we defined $Map_{Set_\Delta}(J,K) \in Set_\Delta$ by the formula
$$Hom_{Set_\Delta}(\Delta^n, Map_{Set_\Delta}(J,K)) \cong Map_{Set_\Delta}(J,K)_n \coloneqq Hom_{Set_\Delta}(\Delta^n \times J, K).$$

Excercise This construction gives Set_Δ the structure of a simplicial category.

Example 20.3. Let $\mathcal{C}$ be a topological category. Then we defined $\Pi(\mathcal{C}) \in Cat_\Delta$ by setting $Ob(\Pi(\mathcal{C})) \coloneqq Ob(\mathcal{C})$ and $Map_{\Pi(\mathcal{C})}(X,Y) \coloneqq \Pi(Map_{\mathcal{C}}(X,Y))$ for all $X, Y \in Ob(\mathcal{C})$. This preserves the composition law since $\Pi(-)$ preserves products.
- We can now relate topological and ∞-categories by defining a simplicial nerve:
$$Cat_{top} \xrightarrow{\Pi(-)} Cat_\Delta \xrightarrow{N(-)} Set_\Delta$$

Definition 20.4. Define a functor $\Delta \overset{C[-]}{\to} Cat_\Delta$ as follows. Given $0 \leq i \leq j$ let P_{ij} denote the partially ordered set

$$P_{ij} = \{I \subset [i,j] \subset \mathbb{N} | i,j \in I\}$$

ordered by inclusions. Then $Ob(C[n]) = \{0, \ldots, n\}$ and for $i, j \in \{0, \ldots, n\}$

$$Map_{C[n]}(i,j) = \begin{cases} \emptyset & i > j \\ N(P_{ij}) & i \leq j \end{cases}$$

For $i \leq j \leq k$ the composition rule

$$Map_{C[n]}(i,j) \times Map_{C[n]}(j,k) \to Map_{C[n]}(i,k)$$

using the natural map of partially ordered sets

$$P_{ij} \times P_{jk} \to P_{ik}$$

given by taking unions.

The simplicial nerves $N(\mathcal{C})$ of a simplicial category is then defined as before:

$$\begin{tikzcd} \Delta^{op} \arrow[r, "{C[-]}"] \arrow[rr, bend right, "N(-)"'] & Cat_\Delta^{op} \arrow[r, "{Hom_{Cat_\Delta}(-,\mathcal{C})}"] & Set \end{tikzcd}$$

Example 20.5. Given $0 \leq i \leq j \leq n$, write $p_{ij} \in Map_{C[n]}(i,j)_0$ for the element corresponding to $\{i,j\} \subset [i,j]$. Then $p_{jk} \circ p_{ij} \neq p_{ik}$ in general, but they are related by a "universal homotopy". For example when $n = 2$, we have

$$P_{02} = \{\{0,2\}, \{0,1,2\}\} = \{p_{02}, p_{12} \circ p_{01}\} \cong ([1] \text{ as a poset})$$

In a picture:

$$\begin{tikzcd} & 1 \arrow[dr, "p_{01}"] \arrow[d, Rightarrow, "{\{0,2\}\subset\{0,1,2\}}"'] & \\ 0 \arrow[ur, "p_{02}"] \arrow[rr, "p_{12}"'] & {} & 2 \end{tikzcd}$$

21. 10/29/2021

Last time: Cat_∞ and Yoneda

Theorem 21.1. *Let $\mathcal{C}$ be an ∞-category.*

(1) There exists a mapping space functor

$$Map_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to Top_\infty$$

compatible with our previous constructions (i.e. $Map_{\mathcal{C}}(X,Y)$ is homotopy equivalent to $Hom^L_{\mathcal{C}}(X,Y)$ for all $X, Y \in Ob(\mathcal{C})$).

(2) (Yoneda) The induced functor $\mathcal{C} \to Fun(\mathcal{C}^{op}, Top_\infty)$ is fully faithful (i.e. it induces homotopy equivalences of mapping spaces.)

Adjoint Functors

Definition 21.2. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be functors between ∞-categories $\mathcal{C}$ and $\mathcal{D}$. Then we say (F,G) are an adjoint pair (or F is a left adjoint of G, or G is a right adjoint of F) if there exists a natural transformation

$$u : Id_{\mathcal{C}} \longrightarrow G \circ F$$

such that for all $X \in Ob(\mathcal{C})$ and $Y \in Ob(\mathcal{D})$, the composition

$$Map_{\mathcal{D}}(F(X), Y) \underset{1}{\rightarrow} Map_{\mathcal{C}}(G(F(X)), G(Y)) \underset{2}{\rightarrow} Map_{\mathcal{C}}(X, G(Y)) \ (*)$$

1: because G is a functor

2: compose with $u_X : X \to G(F(X))$

is a homotopy equivalence.

If $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories, the mapping spaces in $(*)$ are discrete, and we recover the classical definition of adjoint functors.

We call u the unit of the adjunction. One can show there is a counit natural transformation $V : F \circ G \longrightarrow Id_{\mathcal{D}}$ which satisfies a similar property to u.

Theorem 21.3. *Any two right adjoints of F are isomorphic, and the space of isomorphisms compatible with their unit is contractible.*

Theorem 21.4. *Write $LFun(\mathcal{C}, \mathcal{D}) \subset Fun(\mathcal{C}, \mathcal{D})$ for the full subcategory of left adjoints, similarly for $RFun(\mathcal{D}, \mathcal{C}) \subset Fun(\mathcal{D}, \mathcal{C})$. Then there is a canonical equivalence $LFun(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} RFun(\mathcal{D}, \mathcal{C})$ such that the image of any left adjoint is a right adjoint.*

Back to Loops and Suspensions

Recall that a zero object in an ∞-category $\mathcal{C}$ is an object which is both initial and final.

Definition 21.5. Let $\mathcal{C}$ be a pointed ∞-category with zero object $0 \in Ob(\mathcal{C})$. We write $\Omega : \mathcal{C} \to \mathcal{C}$ and $\Sigma : \mathcal{C} \to \mathcal{C}$ for the functors given by

$$\Omega(X) = lim \left(\begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & \longrightarrow & X \end{array} \right) \qquad \Sigma(X) = colim \left(\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array} \right)$$

if the needed limits and colimits exist.

Note: We are using a fact that we have not explicitly proved, which is that limits and colimits are functorial under morphisms of diagrams.

Exercise: If $\mathcal{C}$ is an ordinary category, then $\Omega(X) \equiv 0$ and $\Sigma(X) \equiv 0$ for all $X \in Ob(\mathcal{C})$.

Theorem 21.6. *Let $\mathcal{C}$ be an ∞-category in which Ω and Σ are well-defined. Then (Σ, Ω) is an adjoint pair.*

22. 11/1/2021

Last time: Adjoint functors, loops, and suspensions.

Recall the following two definitions:

Definition 22.1. Functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ between ∞-categories are *adjoint* if there exists functorial isomorphisms

$$Map_{\mathcal{D}}(F(X), Y) \simeq Map_{\mathcal{C}}(X, G(Y))$$

in Top_∞ for all $X \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{D})$ (i.e. isomorphisms induced from a unit or counit transformation).

Definition 22.2. Let $\mathcal{C}$ be a pointed ∞-category. Then, the loop and suspension functors

$$\Omega_{\mathcal{C}}, \Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$

are defined by

$$\Omega_{\mathcal{C}}(X) = \lim \left(\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \right) \qquad \Sigma_{\mathcal{C}}(X) = \operatorname{colim} \left(\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd} \right)$$

provided these limits and colimits exist.

Convenient assumption: call a simplical set *finite* if it has finitely many nondegenerate simplices.

Examples:

$$\Lambda^2_0 = N \left(\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \right) \qquad \Lambda^2_2 = N \left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd} \right)$$

Definition 22.3. We say that an ∞-category $\mathcal{C}$ *admits finite limits* (resp. *colimits*) if any diagram $p : K \to \mathcal{C}$ with K finite has a limit (resp. colimit).

Theorem 22.4. *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits and colimits. Then,*

$$\Sigma_{\mathcal{C}}, \Omega_{\mathcal{C}})$$

are an adjoint pair.

Proof. **(Sketch)**

We claim that any $X, Y \in Ob(\mathcal{C})$, we have isomorphisms:

$$Map_{\mathcal{C}}(\Sigma(X), Y) \simeq \left\{ \text{diagrams} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \right\} \simeq Map_{\mathcal{C}}(X, \Omega(Y))$$

where the middle diagram is in $Fun(\Lambda^1 \times \Lambda^1, \mathcal{C})$.

To see the RHS, consider the following projection maps:

$$\left\{ \text{diagrams} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \right\} \leftarrow \left\{ \text{diagrams} \quad \begin{tikzcd} X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, bend right=15] & & \\ & \Omega(Y) \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ & 0 \arrow[r] & Y \end{tikzcd} \right\} \to \{\text{diagrams } X \to \Omega(Y)\}$$

Both projections are surjective, so (fact from homotopy theory) they are homotopy equivalences if and only if their fibers are contractible.

To see the left arrow, if p is the diagram:

$$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

then $\Omega(Y)$ is, by definition, the final object of $\mathcal{C}/p$. Hence, a diagram $(*)$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is an object of $\mathcal{C}/p$. A diagram as in the middle is a morphism from

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \quad \to \quad \begin{tikzcd} \Omega(Y) \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} \quad \in \mathcal{C}/p$$

So the fiber of the left projection over the diagram $(*)$ is the space of maps as the diagram above. But these are contractible, because the latter is a final object in $\mathcal{C}/p$.

To see the right arrow:

$$\left\{ \text{diagrams} \begin{tikzcd} X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[ddrr, bend right] & & \\ & \Omega(Y) \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ & 0 \arrow[r] & Y \end{tikzcd} \right\} \to \{\text{diagrams } X \to \Omega(Y)\}$$

Given $X \xrightarrow{f} \Omega(Y)$, the fiber of this over f is roughly the extra data needed to factor

$$X \xrightarrow{f} \Omega(Y) \to Y$$

through 0. Because 0 is a zero object, each of the other simplices that make up this data are unique up to contractible choices. $\square$

23. 11/3/2021

24. 11/5/2021

Last Time: Stable ∞-categories, fibers, and cofibers

Example 24.1. Let $F\colon X_\bullet \to Y_\bullet$ be a morphism in Top^*_∞, and let p be the diagram used to define $\mathrm{fib}(f)$. That is, $p\colon \begin{smallmatrix} & \cdot \\ & \downarrow \\ \cdot \to & \cdot \end{smallmatrix} \to \mathrm{Top}^*_\infty$. Then

$$\begin{aligned} \mathrm{Ob}((\mathrm{Top}^*_\infty)_{/p}) &= \left\{ \text{diagrams} \begin{tikzcd} Z \arrow[r, "g"] \arrow[d] \arrow[dr, "h_1", "h_2"'] & X \arrow[d, "f"] \\ 0 \arrow[r, "y_0"'] & Y \end{tikzcd} \right\} \\ &= \Big\{(Z, g, h) \text{ where } g\colon Z \to X \text{ is a pointed map and} \\ &\qquad h\colon Z \times [0,1] \to Y \text{ is such that} \\ &\qquad h|_{Z\times\{0\}} \equiv y_0 \text{ and } h|_{Z\times\{1\}} = f \circ g\Big\} \\ &= \Big\{(Z, g, \bar{h}),\ \text{where } \bar{h}\colon Z \to \mathrm{Maps}([0,1], Y) \\ &\qquad \text{takes } z \in Z \text{ to } h(Z, -)\colon [0,1] \to Y\Big\}, \end{aligned} \tag{24.2}$$

where h is obtained from combining h_1 and h_2. Thus, the universal data of this kind is the subspace of $X \times \mathrm{Maps}([0,1], Y)$ satisfying the needed conditions:

$$\mathrm{fib}(f) = \Big\{(x, \gamma) \in X \times \mathrm{Map}([0,1], Y) \text{ s.t. } \gamma(0) = y_0 and \gamma(1) = f(x)\Big\}.$$

Note that this recovers $\Omega(Y_\bullet)$ when $X = pt$.

Aside: When there is potential for ambiguity we call $\mathrm{fib}(f)$ the ***homotopy*** fiber of f. The ***set-theoretic*** fiber of f is just $f^{-1}(y_0) \subset X_\bullet$. When the set-theoretic fibers of f don't change too wildly (technically: when f is a Serre fibration) the set-theoretic fiber is weakly homotopy equivalent to the homotopy fiber.

We can also characterize stable ∞-categories in terms of how fibers and cofibers interact.

Definition 24.3. Let $\mathcal{C}$ be a pointed ∞-category. A ***triangle*** in $\mathcal{C}$ is a diagram $\Lambda^1 \times \Lambda^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] \arrow[dr, "\phi" near start, "h_1", "h_2"'] & X \arrow[d, "f"] \\ 0 \arrow[r, "y_0"'] & Y \end{tikzcd}$$

That is, it is the data of a pair of morphisms f and g, a 2-simplex h_1, identifying ϕ as a composition of f and g, and a 2-simplex h_2 factoring ϕ through 0 (a ***null homotopy*** of ϕ). We'll often write this data as $X \xrightarrow{f} Y \xrightarrow{g} Z$ leaving (ϕ, h_1, h_2) implicit.

Example 24.4. If $\mathcal{C}$ is a pointed ordinary category, then (ϕ, h_1, h_2) are determined by f and g, and f and g define a triangle iff $g \circ f = 0$.

Note: Consider the composition $\mathrm{Map}_{\mathcal{C}}(X, 0) \times \mathrm{Map}_{\mathcal{C}}(0, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$. The domain is contractible, so we can talk about "the zero morphism from X to Z", understanding that this is well-defined up to a contractible space of choices.

Definition 24.5. We say a triangle is a ***fiber sequence*** if it is a pullback diagram (so $X = \mathrm{fib}(g)$) and a ***cofiber sequence*** if it is a pushout diagram (so $Z = \mathrm{cof}(f)$).

Example 24.6. If $\mathcal{C}$ is a pointed ordinary category then a triangle is a fiber sequence (resp. cofiber) iff $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is ***left exact*** (resp. right exact).

Theorem 24.7. *An ∞-category is stable iff*

i) it is pointed
ii) every morphism admits a fiber and cofiber
iii) every fiber sequence is a cofiber sequence, and every cofiber sequence is a fiber sequence.

25. 11/15/2021

Last time: $\mathcal{C}$ stable $\Rightarrow Ho(\mathcal{C})$ additive

Given $X, Y \in Ob(\mathcal{C})$ we can see the group structure on $Hom_{Ho(\mathcal{C})}(X, Y)$ more explicitly by recalling that

$$Map_{\mathcal{C}}(X,Y) \simeq Map_{\mathcal{C}}(X, \Omega\Sigma Y) \simeq \left\{ \text{diagrams in } \mathcal{C} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_2", "h_1"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd} \right\}$$

where $\simeq$ represents homotopy equivalence.

We can paste two such diagrams together

$$\left(\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_2", "h_1"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd} , \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_2'", "h_1'"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd} \right) \mapsto \begin{tikzcd} X \arrow[rr] \arrow[dd] \arrow[dr] \arrow[ddrr, bend left=40] \arrow[ddrr, bend right=40] \arrow[dr, phantom, shift left=3ex, "h_1'"] \arrow[dr, phantom, shift right=3ex, "h_1"] & & 0 \arrow[dd] \arrow[ddl, phantom, "h_2'"] \\ & 0 \arrow[dr] \arrow[l, phantom, "h_2"] & \\ 0 \arrow[rr] & & \Sigma Y \end{tikzcd}$$

(really we should be careful about the fact that the morphism $X \to 0$ is only unique in a homotopical sense).

One can "replace" the upper right part with a single 2-simplex h_2" to get a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_2\text{''}", "h_1"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$

where h_2' and h_2" have identical top and right edges (i.e. define h_2" using horn-filling properties of ∞-categories and properties of zero objects).

This replacement is not unique, but different replacements are connected by paths in $Map_{\mathcal{C}}(X, Y)$. Hence the resulting binary operation on $Hom_{Ho(\mathcal{C})}(X, Y)$ is well-defined.

Another important feature of stable ∞-categories is that we can "rotate triangles".

Proposition 25.1. *Let $\mathcal{C}$ be a stable ∞-category and*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

a fiber sequence in $\mathcal{C}$. Then $cof(g) \simeq \Sigma X \simeq \Sigma(fib(g))$. In other words, there exists a fiber sequence of the form

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

(these two sequences differ by a "rotation"). Thus in a stable ∞-category fibers and cofibers are closely related (contrast this with how things work in an abelian category).

Corollary 25.2. *There exists a fiber sequence of the form*

$$Z \xrightarrow{h} \Sigma X \xrightarrow{*} \Sigma Y$$

(in fact this morphism (*) is $-\Sigma f$, where "minus" refers to the additive group structure).

Remark 25.3. Our discussion of the group structure on $Hom_{Ho(\mathcal{C})}(X, Y)$ via "diagram pasting" can be extended to prove that reversed diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_2", "h_1"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, "h_1", "h_2"'] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$

correspond to inverse elements of $Hom_{Ho(\mathcal{C})}(X, Y)$ with respect to the abelian group structure.

26. 11/19/2021

Last time we talked about rotation of triangles and generalized the notion of **exact sequence** from **Abelian category** to fiber sequence in stable $\infty-$category. From the rotation we have that cofiber and fiber are quite the same object, which is different from **Abelian category**.

Definition 26.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty-$categories with finite limits and colimits. Then $F : \mathcal{C} \to \mathcal{D}$ is left (resp. right) exact if it preserves finite limits (resp. colimits).

Theorem 26.2. *If $\mathcal{C}$ and $\mathcal{D}$ are stable $\infty-$categories, then $F : \mathcal{C} \to \mathcal{D}$ is left exact if and only if it is right exact.*

Proof. Recall from last time that F is left exact $\iff$ F preserves direct sums and fibers and that F is right exact $\iff$ F preserves direct sums and cofibers. Hence we just need to show that if F is left exact, then it preserves cofibers, and if F is right exact, then it preserves fibers.

Suppose that F is left exact, and let $g : X \to Y$ be a morphism in $\mathcal{C}$. Note that F preserves Ω since it is left exact, that is, $F(\Omega Z) = \Omega F(Z)$. Hence, it also preserves $[n]$ for any n. Then

$$\begin{aligned} F(cof(g)) &\simeq F(\Sigma fib(g)) \\ &\simeq \Sigma F(fib(g)) \\ &\simeq \Sigma fib(F(g)) \\ &\simeq cof(F(g)) \end{aligned}$$

By a dual argument, F preserves fibers if it is right exact. □

Summary of last few lectures:

(1) Stable $\infty-$categories have a number of distinguished features:
   (a) they have an intrinsic $\mathbb{Z}-$symmetry $X \to X[n]$
   (b) they can be characterized in a similar (but simpler way) as Abelian categories
   (c) they have additive homotopy categories
   (d) right exact functors between them are left exact, and vice versa.

   A natural question arises: are there actual examples? The answer is Yes. This is because there are universal constructions for producing stable $\infty-$categories from non-stable $\infty-$categories.

(2) We will study a few such constructions, starting with the **Spanier-Whitehead category** $SW(\mathcal{C})$ of a pointed $\infty-$category $\mathcal{C}$ with finite colimits (a.k.a, the category of $\Sigma-$spectrum objects of $\mathcal{C}$).

Analogy: Consider $\mathbb{Z}[x]$ together with the multiplication operator $x : \mathbb{Z}[x] \to \mathbb{Z}[x]$ that takes $f(x)$ to $xf(x)$, which is a homomorphism of $\mathbb{Z}[x]-$modules as Abelian groups. This operator is not invertible, but there's a universal construction which makes it invertible: the localization $\mathbb{Z}[x, x^{-1}]$. If we analogize $(\mathcal{C}, \Sigma_{\mathcal{C}})$ with $(\mathbb{Z}[x], x)$, is there a corresponding analogue of $\mathbb{Z}[x, x^{-1}]$?

Note that we have a filtration

$$\mathbb{Z}[x] \subset x^{-1}\mathbb{Z}[x] \subset x^{-2}\mathbb{Z}[x] \subset \cdots \subset \mathbb{Z}[x, x^{-1}].$$

In other words, $\mathbb{Z}[x, x^{-1}] \simeq \operatorname{colim}_{n\in\mathbb{N}} x^{-n}\mathbb{Z}[x]$ as $\mathbb{Z}[x]-$modules as Abelian groups. Note that as $\mathbb{Z}[x]-$modules, each $x^{-n}\mathbb{Z}[x]$ is isomorphic to $\mathbb{Z}[x]$ itself.

27. 11/29/2021

Last time: the Spanier-Whitehead category.

Definition 27.1. Let $\mathcal{C}$ be a pointed ∞-category with finite colimits. Then

$$SW(\mathcal{C}) := \operatorname{colim}_{\mathbb{N}}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \cdots)$$

where the colimit is taken in Cat_∞.

- Explicitly, we have a functor $i_n : \mathcal{C} \to SW(\mathcal{C})$ for every $n \in \mathbb{N}$, and every object of $SW(\mathcal{C})$ is in the essential image of i_n for all $n >> 0$.
- This corresponds to the fact that every Laurent polynomial is in the image of

$$\mathbb{Z}[x] \to \mathbb{Z}[x, x^{-1}]$$

$$f(x) \mapsto x^{-n} f(x)$$

for $n >> 0$.
- Given $m, n \in \mathbb{N}$ and $X, Y \in Ob(\mathcal{C})$, we moreover have

$$Map_{SW(\mathcal{C})}(i_n(X), i_n(Y)) \simeq \operatorname*{colim}_{k \geq m,m} Map_{\mathcal{C}}(\Sigma^{k-m} X, \Sigma^{k-n} Y)$$

where the colimit is taken in Top_{∞}^{*}.
- We also have that

$$Hom_{Ho(SW(\mathcal{C}))}(i_n(X), i_n(Y)) \simeq \operatorname*{colim}_{k \geq m,m} Hom_{Ho(\mathcal{C})}(\Sigma^{k-m} X, \Sigma^{k-n} Y)$$

where the colimit is taken in Set. ($Rightarrow$ we can describe $Ho(SW(\mathcal{C}))$ purely in terms of $Ho(\mathcal{C})$)

Theorem 27.2. *(1) $SW(\mathcal{C})$ is stable.*

(2) Let $\mathcal{D}$ be any stable ∞-category. Then composition with $i_0 : \mathcal{C} \to SW(\mathcal{C})$ induces an equivalence

$$Fun^{rex}(SW(\mathcal{C}), \mathcal{D}) \simeq Fun^{rex}(\mathcal{C}, \mathcal{D})$$

between the categoryies of right exact functors from $\mathcal{C}$ and $SW(\mathcal{C})$ to $\mathcal{D}$.

- Recall that the universal property of $x^{-1}M$ was characterized as follows: if N is any $\mathbb{Z}[x]$-module on which x acts invertibly, composition with $M \xrightarrow{\varphi_x} x^{-1}M$ induces a bijection

$$Hom_{\mathbb{Z}[x]-mod}(x^{-1}M, N) \simeq Hom_{\mathbb{Z}[x]-mod}(M, N)$$

between the sets of $\mathbb{Z}[x]$-module homomorphisms. So (2) says that swc is characterized by a universal property just like $x^{-1}M$ is.

Proof. (1) We can describe $\Sigma_{SW(\mathcal{C})}$ and its inverse via maps of directed systems:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "\Sigma_{\mathcal{C}}"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "\Sigma_{\mathcal{C}}"'] & \mathcal{C} \arrow[r] \arrow[d, "\Sigma_{\mathcal{C}}"'] & \cdots \\
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[ur, "id_{\mathcal{C}}"] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[ur, "id_{\mathcal{C}}"] & \mathcal{C} \arrow[r] \arrow[ur, "id_{\mathcal{C}}"] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}
SW(\mathcal{C}) \arrow[d, shift right, "\Sigma_{SW(\mathcal{C})}"'] \\
SW(\mathcal{C}) \arrow[u, shift right, "\Omega_{SW(\mathcal{C})}"']
\end{tikzcd}$$

By construction the endomorphism of swc induced by the red arrow is inverse to the one induced by the blue arrow, which is just $\Sigma_{SW(\mathcal{C})}$.

(2) Note that $\mathcal{D} \simeq SW(\mathcal{D})$ since $\mathcal{D}$ is stable. Then the inverse of composition with i_0 take a right-exact functor $F : \mathcal{C} \to \mathcal{D}$ to the functor $\tilde{F} : SW(\mathcal{C}) \to \mathcal{D}$ defined by

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "\Sigma_{\mathcal{C}}"] \arrow[d, "F"'] & \cdots & SW(\mathcal{C}) \arrow[d, "\tilde{F}"] \\
\mathcal{D} \arrow[r, "\Sigma_{\mathcal{D}}"] & \mathcal{D} \arrow[r, "\Sigma_{\mathcal{D}}"] & \mathcal{D} \arrow[r, "\Sigma_{\mathcal{D}}"] & \cdots & SW(\mathcal{D})
\end{tikzcd}$$

(Note that since swc and sD are stable, the extension $\tilde{F} : SW(\mathcal{C}) \to \mathcal{D}$ is aotomatically left exact even if F is not.)

Bad example: if $\mathcal{C}$ is an ordinary abelian category, then $\Sigma_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the zero functor, i.e. $\Sigma_{\mathcal{C}}(X) \simeq 0$ for all $X \in Ob(\mathcal{C})$, hence $SW(\mathcal{C})$ is just the zero category, i.e. it is contractible.