LAWRGE 2024 NOTES

The following notes were taken and compiled by the participants of the 2024 Los Angeles Workshop on Representations and Geometry. The workshop consisted of lectures by David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh on their work "Relative Langlands Duality."

Disclaimer: This document reflects the participants' heroic efforts to capture everything that was said and written during the lectures, but nonetheless there are many, many omissions, and the result should be understood as an extremely noisy approximation of the live event. Moreover, these notes have not been proofread or edited in any systematic fashion, and any mistakes or errors should be attributed to noise in the note-taking process rather than to the speakers.

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1. Day 1 Lecture 1: Introduction to Topological Quantum Field Theory

Topological quantum field theory (TQFT) serves as a very useful metaphor in much of representation theory. Many questions in representation theory can be organized in terms of TQFTs. In today's lecture, we will consider toy models associated to a finite group G. In what follows, to any finite group G we associate an n-dimensional TQFT which we will denote by Z_G^n, or simply Z if the group and dimension are clear from context.

1.1. What is a TQFT?.

A TQFT is a linear representation of the bordism category Bord^n of manifolds of dimension $\leq n$. By a linear representation we mean that we want a TQFT to be a functor out of Bord^n into a category which in some ways resembles the category of vector spaces (over $\mathbb{C}$, say). For example, this functor assigns

(i) to a closed n-dimensional manifold Σ^n a number $Z(\Sigma^n) \in \mathbb{C}$,
(ii) to an $(n-1)$-dimensional manifold Σ^{n-1} a vector space $Z(\Sigma^{n-1}) \in \mathrm{Vect}_{\mathbb{C}}$,
(iii) to an $(n-2)$-dimensional manifold Σ^{n-2} a $\mathbb{C}$-linear category $Z(\Sigma^{n-2}) \in \mathrm{Cat}_{\mathbb{C}}$,

and so on. As the codimension of Σ increases, so too should the categorical complexity of $Z(\Sigma)$.

What properties do we want this functor to satisfy?

(i) We want this functor to be symmetric monoidal, where the monoidal structure in Bord^n is given by disjoint union $\sqcup$, and in the target category by tensor product $\otimes$. A TQFT will send the empty manifold ϕ_m of dimension m to the monoidal unit in the corresponding category:

$$Z(\phi_n) = 1 \in \mathbb{C}, \qquad Z(\phi_{n-1}) = \mathbb{C} \in \text{Vect}_{\mathbb{C}} \ldots$$

This property corresponds to the Q in TQFT and can be thought of a mathematical encoding of superposition of quantum states.

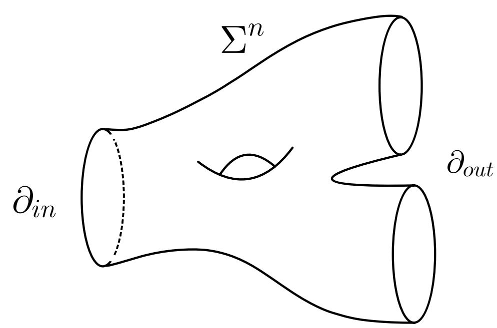
(ii) We want this functor to be locally constant in families. That is, $Z(\Sigma^n)$ should be a locally constant function on the moduli space of n-dimensional manifolds. Similarly, $Z(\Sigma^{n-1})$ should be a locally constant sheaf over this moduli space, which carries an action of $\text{Diff}(\Sigma^{n-1})$. This corresponds to the first 'T' in TQFT.

Remark 1.1. For our purposes, we can think of the moduli space of n-manifolds as $\coprod_{M^n} \mathbf{B}\text{Diff}(M)$, where the coproduct is taken over all n-manifolds.

1.2. **Morphisms.** A morphism between two $(n-1)$-dimensional manifolds ∂_{in} and ∂_{out} in Bord^n is an n-dimensional manifold Σ such that $\partial\Sigma = \partial_{in} \sqcup \partial_{out}$.

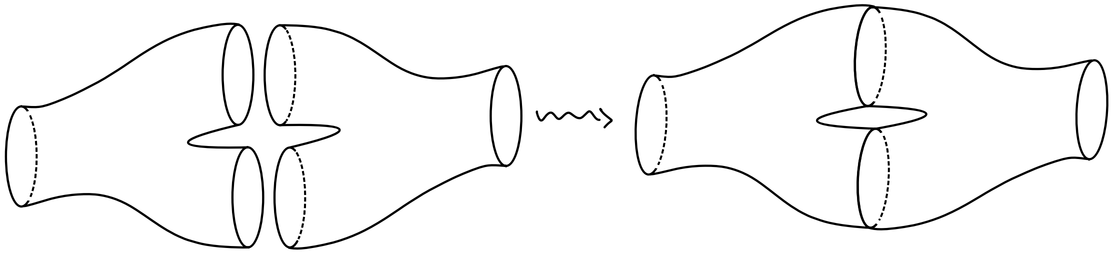
Remark 1.2. We are assuming that our TQFTs are *oriented*. This is what enables us to distinguish between incoming and outgoing boundaries.

We denote such a morphism symbolically by $\partial_{in} \xrightarrow{\Sigma^n} \partial_{out}$ and pictorially by

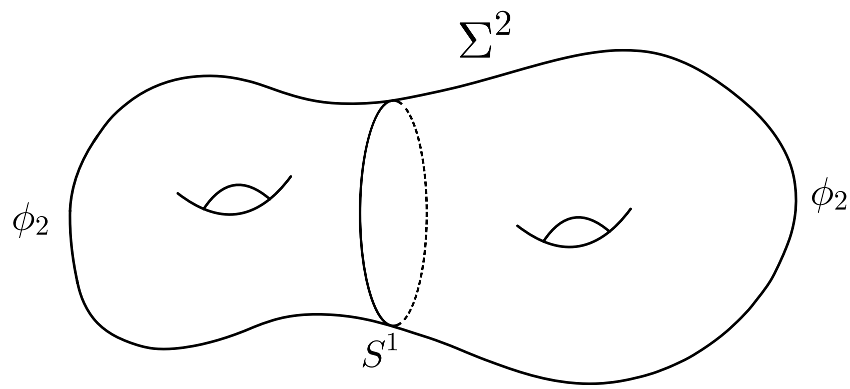


Since Z is a functor, there is an associated morphism $Z(\partial_{in}) \xrightarrow{Z(\Sigma^n)} Z(\partial_{out})$ which is given by a linear transformation. Composition of morphisms corresponds to gluing bordisms together, which can be represented pictorially as

Physically, this is meant to encode the idea of *locality*, that the physics at a point is entirely determined by what happens near it. This allows one to compute the output of Z on a



complicated manifold by chopping it up into more manageable pieces. For example, we can consider breaking up a surface as follows



from which we see that we can compute $Z(\Sigma^2)$ as a composition $\mathbb{C} \to Z(S^1) \to \mathbb{C}$.

Example 1.3 (1d-Oriented TQFT and Beyond)**.** The functor Z assigns to a point with positive orientation a vector space $Z(\bullet^+) = V$, and to a point with negative orientation another vector space $Z(\bullet^-) = V^\vee$.

(i) There are natural maps $\mathbb{C} \to V \otimes V^\vee$ and $V \otimes V^\vee \to \mathbb{C}$. One can show that by considering the composition $\mathbb{C} \to V \otimes V^\vee \to \mathbb{C}$ that, in fact, $V^\vee = V^*$ and the second map $V \otimes V^\vee \to \mathbb{C}$ is given by evaluation.

(ii) We find that $Z(S^1) = \dim V$, and more generally that in an n-dimensional theory we have $Z(\Sigma^{n-1} \times S^1) = \dim Z(\Sigma^{n-1})$.

(iii) Given a diffeomorphism $f : \Sigma^{n-1} \to \Sigma^{n-1}$, there is a corresponding linear map $Z(f) : Z(\Sigma^{n-1}) \to Z(\Sigma^{n-1})$. We can see that $\operatorname{tr} Z(f)$ corresponds to Z applied to the mapping torus of f (an n-manifold).

1.3. **Where are the Fields?** In order to talk about quantum field theory, all that is really required is a way to map from physical spacetimes to spaces of states. This is what the functor Z accomplishes. However, in many cases of interest, Z factors though a geometric category, as in the following diagram:

$$\begin{tikzcd}
\mathrm{Bord}^n \arrow[rr, "Z"] \arrow[dr, "\mathcal{F}_Z"'] & & \text{Linear categories} \\
& \text{Spaces (stacks) with correspondences} \arrow[ur, "G"'] &
\end{tikzcd}$$

This type of factorization is additional structure on the TQFT, and a TQFT together with such a factorization is called a *Lagrangian TQFT*. The functor $\mathcal{F}_Z$ in the diagram assigns spaces of *fields* to objects of Bord^n. Typically, this space is given by $\mathrm{Map}(-, T)$ where T is some target manifold, and G is given by taking functions or categories of sheaves on this space.

Remark 1.4. Most TQFTs that you will find in physics textbooks are Lagrangian, but there are important theories which do not have known Lagrangian structures. Moreover, the same TQFT can have multiple Lagrangian structures, all of which result in the same physical theory. This can result in surprising mathematical statements or dualities in physics.

Example 1.5. Let G be a finite group and consider the corresponding groupoid BG; i.e. BG is a category with a single object $*$ and morphisms $\mathrm{Hom}(*, *) = G$.

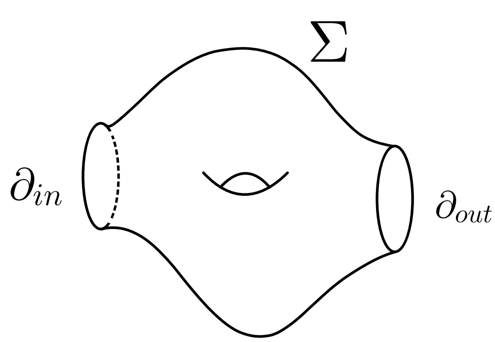
(i) There exists a Lagrangian TQFT with space of fields given by $\mathcal{F}_G(\Sigma) = \mathrm{Map}(\Sigma, BG)$, which is also called the space of local systems on G, denoted $\mathrm{Loc}_G\Sigma$. Note that this is what the TQFT assigns to Σ, regardless of its dimension.

(ii) This space can equivalently be thought of as the space of G-bundles on Σ, the space of G-covering spaces, and as the space of maps $\pi_1(\Sigma) \to G$ up to conjugation.

Remark 1.6. The identification of the stack of G-bundles with G-local systems on Σ fails when G is not a discrete group.

(iii) This TQFT assigns to S^1 the points of G modulo the conjugation action. So fields on the circle are given by group elements up to conjugation.

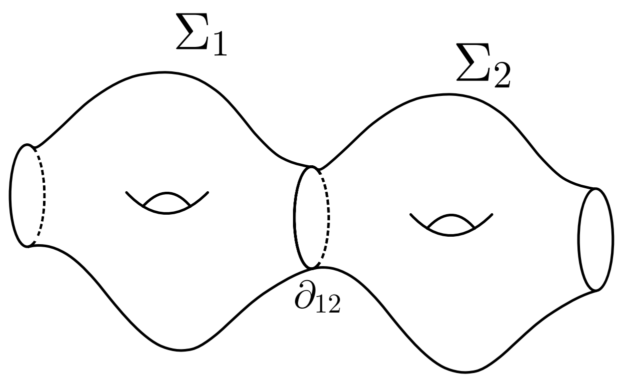
Given a bordism $\partial_{in} \xrightarrow{\Sigma} \partial_{out}$, the functor $\mathcal{F}_G$ yields a correspondence through restriction. That is, if we consider the bordism



then from this we obtain the following maps by restricting to the boundary components

$$\begin{tikzcd} & \mathcal{F}_G(\Sigma) \arrow[dl] \arrow[dr] & \\ \mathcal{F}_G(\partial_{in}) & & \mathcal{F}_G(\partial_{out}) \end{tikzcd}$$

Similarly, if we glue two bordisms together we obtain the picture



which corresponds to the following diagram.

$$\begin{tikzcd} & & \mathcal{F}_G(\Sigma_1 \sqcup_{\partial_{12}} \Sigma_2) \arrow[dl] \arrow[dr] & & \\ & \mathcal{F}_G(\Sigma_1) \arrow[dl] \arrow[dr] & & \mathcal{F}_G(\Sigma_2) \arrow[dl] \arrow[dr] & \\ \mathcal{F}_G(\partial_1) & & \mathcal{F}_G(\partial_{12}) & & \mathcal{F}_G(\partial_2) \end{tikzcd}$$

Similarly, given a 2-manifold Σ^2, we have

$$\mathcal{F}_G(\Sigma) = \mathrm{Loc}_G(\Sigma) = \mathrm{Maps}(\pi_1(\Sigma), G)/\mathrm{Conjugation},$$

which is the G-character stack of Σ.

1.4. **Linearizing the Space of Fields.** Continuing the example above where $\mathcal{F}_G = \mathrm{Hom}(-, BG)$, in order to construct a TQFT, we need to linearize the spaces of fields. While $\mathcal{F}_G(\Sigma)$ did not depend on the dimension of Σ, our linearization will. Let X be a finite orbifold, such as $\mathrm{Loc}_G(\Sigma)$. If Σ is of top dimension, then we attach a number to $X = \mathrm{Loc}_G(\Sigma)$ by

$$\#X = \int_X 1 = \sum_{x \in X} \frac{1}{|\mathrm{Aut}\, x|}$$

If Σ has codimension one, then to $X = \mathrm{Loc}_G(\Sigma)$ we attach the vector space of global sections of the constant sheaf, i.e.,

$$\mathbb{C}[X] = \Gamma(X, \underline{\mathbb{C}}) = \bigoplus_{x \in X} \mathbb{C}$$

For codimension two Σ, we assign the category of vector bundles on X:

$$\mathrm{Vect}[X] = \bigoplus_{x \in X} \mathrm{Vect}\left(\bullet/\mathrm{Aut}\, x\right)$$

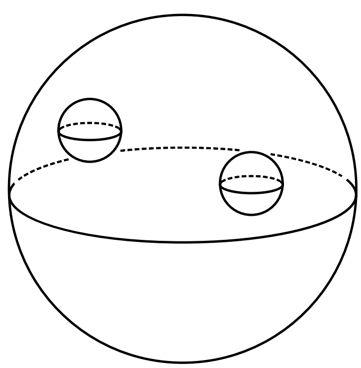
where $\mathrm{Vect}\left(\bullet/\mathrm{Aut}\, x\right)$ by definition is the category of representations of this automorphism group.

Example 1.7. Fix a finite group G. Then, we can consider what different dimensions of TQFT assign to different dimensions of manifold. The empty sections are not of interest for now.

	Z_G^2	Z_G^3	Z_G^4
Ξ^3	N/A	$\mathrm{Loc}_G\Xi$	$\mathbb{C}[\mathrm{Loc}_G\Xi]$
Σ^2	$\#\mathrm{Loc}_G\Sigma$	$\mathbb{C}[\mathrm{Loc}_G\Sigma]$	$\mathrm{Vect}[\mathrm{Loc}_G\Sigma]$
S^1	$\mathbb{C}[G/G]$	$\mathrm{Vect}[G/G]$	
$\bullet$	$\mathrm{Rep}G$		

1.5. **Observables in TQFT.** An observable is a physical quantity we can measure at a point. At any given point, there is an entire algebra of local observables. The data of a local observable can be captured by the state on a sphere surrounding the chosen point. Hence, the algebra of local observables in a TQFT is the space of states on a sphere, and $Z(S^{n-1})$ plays a special role in TQFTs. In particular, this class of observables has several nice algebraic structures.

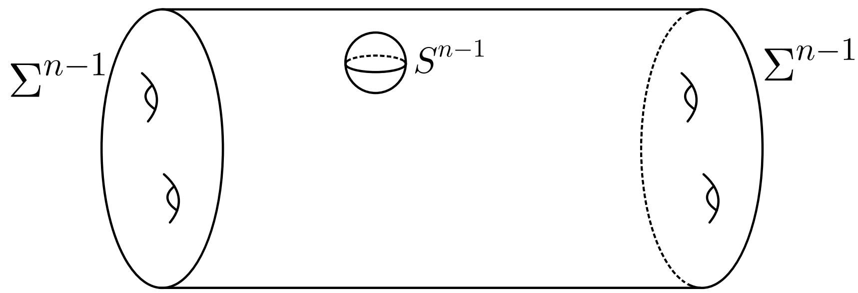
(1) **Algebra structure:** by taking a sphere and removing two smaller spheres from the inside, we obtain a bordism $S^{n-1} \sqcup S^{n-1} \to S^{n-1}$.



This gives a map on the space of observables $Z(S^{n-1}) \otimes Z(S^{n-1}) \to Z(S^{n-1})$. This is a product which is locally constant over the configuration space of two balls inside a larger ball. An algebra with multiplication parameterized by the space of such configurations is called an E_n-algebra.

Note that in dimensions ≥ 2 this multiplication is commutative[1]; however, in 1 dimension we only get an associative algebra.

(2) **Module structure:** Given an $(n-1)$-dimensional manifold Σ^{n-1}, consider the n-dimensional manifold given by $\Sigma^{n-1} \times [0,1]$. From this manifold we can remove a ball S^{n-1} from the middle, thus giving us a bordism $\Sigma^{n-1} \sqcup S^{n-1} \to \Sigma^{n-1}$.



This gives an action map $Z(S^{n-1}) \otimes Z(\Sigma^{n-1}) \to Z(\Sigma^{n-1})$. Thus, the observables on a sphere act on the observables on any other $(n-1)$-dimensional manifold.

Example 1.8. (i) In 2 dimensions, $Z(S^1)$ has a commutative algebra structure, a unit, and a trace operation. These operations make $Z(S^1)$ into a commutative Frobenius algebra. The unit and trace are represented by the pictures

(ii) In our finite group example, this Frobenius algebra is the algebra of class functions on the group: $\mathbb{C}[G/G]$. In this case, the trace is calculated by evaluating at 1 and dividing by $|G|$.

2. Day 1 Lecture 2

Written by: Hyun Jong Kim, Han Li The idea of **arithmetic topology**: basic arithmetic objects, including rings and, should behave like low-dimensional manifolds. The following table lists out arithmetic objects on the left column and analogous topological objects on the right column:

[1] This is not really true, but is sufficient for our purposes. There is a precise sense in which E_n-algebras become more commutative as $n \to \infty$, with the limiting object, an E_∞-algebra, being truly commutative.

$\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{F}_p[i]$, curve X over $\mathbb{F}_p$	3-manifolds
$\mathbb{Z}[\frac{1}{6}]$, $\mathbb{Z}_p$	3-manifolds with boundaries
$\mathbb{Q}_p$, $\mathbb{F}_p(\!(t)\!)$, $\mathbb{R}$, $\mathbb{C}$, curve X over $\overline{\mathbb{F}_p}$	2-manifolds
$\overline{\mathbb{F}_p}(\!(t)\!)$	1-manifolds
ideals	links
prime ideals	knots
$\mathcal{O}_\mathfrak{p}$	tubular neighborhood of $\mathfrak{p}$

A motivation of these analogies is the analogy between π_1(manifold) and the étale fundamental groups of schemes.

History. Around 1880, Dedekind was unhappy with Riemann's (lack of a precise) definition of Riemann surfaces. Consequently he and Weber wrote out a text including a definition of Riemann surfaces. He saw that a branched cover $\Sigma \to \Sigma'$ of Riemann surfaces are analogous to an extension L/K of number fields .

In 1949, Weil conjectured the existence of a purely algebraic approach to studying the topology of algebraic varieties, i.e., for a complex algebraic variety $X = \operatorname{Spec} R$, there should be a way to compute $H^*(X(\mathbb{C}))$ from R alone. In particular, $\mathbb{C}$-automorphisms of R, including insane ones (arising from the arithmetic of $\mathbb{C}$ rather than the geometry of X), should induce a $\mathbb{C}$-automorphism of $H^*(X)$. This was realized by Grothendieck's school with the étale topology $H^*_{\text{ét}}(R)$.

In the 1960s, Poitou and Tate computed $H^*_{\text{ét}}(R)$ for $R = \mathbb{Z}[\frac{1}{p}]$ etc. Mumford and Mazur proposed that $\operatorname{Spec}\mathbb{Z}$ is like a 3-manifold.

Example. Let K be a number field, S be a finite set of places including all Archimedean ones, and $\mathcal{O}$ be the ring of integers in K; alternatively, let $K = X(\mathbb{F}_q)$, X a smooth projective curve, and $\mathcal{O}$ be the ring of functions that are "regular away from S". For example, $K = \mathbb{Q}$, $S = \{\infty, 2\}$, and $\mathcal{O} = \mathbb{Z}[\frac{1}{2}]$. Let M be a coefficient system.

Then there is a long exact sequence:

$$\begin{aligned} \cdots \to H^{3-i}_{\text{ét}}(\operatorname{Spec}\mathcal{O}, M^*)^* \to H^i_{\text{ét}}(\operatorname{Spec}\mathcal{O}; M) \to \bigoplus_{\nu \in S} H^i_{\text{ét}}(\operatorname{Spec} K_\nu; M) \\ \to H^{3-(i+1)}_{\text{ét}}(\operatorname{Spec}\mathcal{O}, M^*)^* \to \cdots \end{aligned} \tag{2.1}$$

(If M is finite, then $M^* = \operatorname{Hom}(M, S^1)$. In general the twist involved in $*$ corresponds to the non-orientability of the manifold.)

This is analogous to the long exact sequence of cohomology, with X a 3-manifold with boundary Y:

$$\cdots \to H^i(X, Y; M) \to H^i(X; M) \to H^i(Y; M) \to \cdots. \tag{2.2}$$

FIGURE 2.1. Spec $\mathcal{O}$ is analogous to a 3-manifold whose boundary is $\bigsqcup_{\nu \in S} \operatorname{Spec} K_\nu$

Under Poincaré duality, which states that $H^i(X, Y; M) \cong H^{3-i}(X; M^*)^*$, the above long exact sequence (2.2) becomes

$$\cdots \to H^{3-i}(X; M^*)^* \to H^i(X; M) \to H^i(Y; M) \to \cdots.$$

which is analogous to (2.1). Notice the absence of Y on the right: X "knows" what ∂X is. This is the **Poincaré–Lefschetz duality**.

Then the analogy suggests that Spec $\mathcal{O}$ is like a (non-orientable) 3-manifold whose boundary is the union $\bigsqcup_{\nu \in S} \operatorname{Spec} K_\nu$, see Figure 2.1.

Of course, as a topological space, Spec $\mathcal{O}$ is not a 3-dimensional space. More generally, we emphasize that the geometric analogies that we record in these notes are analogies and we are not necessarily stating such analogies as precise facts about schemes or spaces.

In the example of $\mathcal{O} = \mathbb{Z}[\frac{1}{2}]$, $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ has two boundary components, $\operatorname{Spec} \mathbb{R}$ and $\operatorname{Spec} \mathbb{Q}_2$, see Figure 2.2. Moreover, a prime ideal is like a knot/cycle — see Figure 2.3, which draws the prime ideal (3) of $\mathbb{Z}[\frac{1}{2}]$ drawn as a knot/cycle.

One can obtain $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ from $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ by cutting out a tube around the knot associated to the prime ideal $3\mathbb{Z}$, i.e., the solid tube $\operatorname{Spec} \mathbb{Z}_3$ with boundary $\operatorname{Spec} \mathbb{Q}_3$, see Figure 2.4; in other words, gluing $\operatorname{Spec} \mathbb{Z}_3$ to $\operatorname{Spec} \mathbb{Z}[\frac{1}{6}]$ along $\operatorname{Spec} \mathbb{Q}_3$ yields $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$.

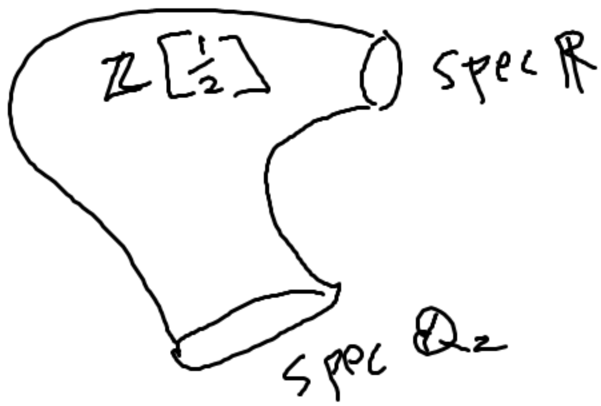


FIGURE 2.2. $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ is analogous to a 3-manifold whose boundary is $\operatorname{Spec}\mathbb{Q}_2 \sqcup \operatorname{Spec}\mathbb{R}$

Langlands as a 4d TQFT. There are no 4-manifolds in the dictionary, so we start by assigning vector spaces to arithmetic objects analogous to 3-manifolds:

$$\begin{aligned} \text{“3-manifolds”} &\mapsto \text{vector spaces} \\ \text{“2-manifolds”} &\mapsto \text{categories} \\ \text{“1-manifolds”} &\mapsto \text{2-categories} \end{aligned}$$

For example, for $G = \mathrm{SL}_n$ `(Question: does anything below change if` G `isn't` SL_n`?)`, the “TQFT” $\mathcal{A}_G$ works as follows:

$$\begin{aligned} &\text{3d}\begin{cases} \mathbb{Z} & \mapsto \text{functions on } G(\mathbb{Z})\backslash G(\mathbb{R}), \text{ i.e., automorphic forms} \\ \mathbb{Z}[\frac{1}{p}] & \mapsto \text{functions on } G(\mathbb{Z}[\frac{1}{p}])\backslash(G(\mathbb{R})\times G(\mathbb{Q}_p)) \\ \text{curve } X \text{ over } \mathbb{F}_p & \mapsto \text{functions on } \operatorname{Bun}_G X \end{cases} \\ &\text{2d}\begin{cases} \mathbb{R} \text{ or } \mathbb{Q}_p & \mapsto \operatorname{Rep} G(\mathbb{R}) \text{ or } \operatorname{Rep} G(\mathbb{Q}_p) \text{ (local Langlands)} \\ \text{curve } X \text{ over } \overline{\mathbb{F}_p} & \mapsto \operatorname{Shv}(\operatorname{Bun}_G X) \text{ (global geometric Langlands)} \end{cases} \end{aligned}$$

On the bordism side, $\partial\mathbb{Z}[\frac{1}{p}] = \mathbb{R} \sqcup \mathbb{Q}_p$; on the algebra side, $\{\text{functions on } G(\mathbb{Z}[\frac{1}{p}])\backslash(G(\mathbb{R}) \times G(\mathbb{Q}_p))\} \in \operatorname{Rep}(G(\mathbb{R}) \times G(\mathbb{Q}_p))$ (the regular representation). This agrees with how manifolds with boundaries work in a 4d TQFT: a 3-manifold with boundary should be mapped to a category together with an object.

As for composition, since $\operatorname{Spec}\mathbb{Z}[\frac{1}{p}] \cup_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec}\mathbb{Z}_p = \operatorname{Spec}\mathbb{Z}$, we expect $\mathcal{A}_G(\mathbb{Z}) = \operatorname{Hom}_{\operatorname{Rep} G(\mathbb{Q}_p)}(\mathcal{A}_G(\mathbb{Z}_p), \mathcal{A}_G(\mathbb{Z}[\frac{1}{p}]))$ (or the other way, but that turns out to not work). This forces $\mathcal{A}_G(\mathbb{Z}_p)$ to be $\operatorname{ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)} \mathbf{1}$ (by definition, $\operatorname{ind}_H^G \rho = \{\phi : G \to V : \phi(gh^{-1}) = \rho(h)\phi(g);$ ϕ has compact support mod $H\}$).

This also shows that $\operatorname{End}_{\operatorname{Rep} G(\mathbb{Q}_p)} \mathcal{A}_G(\mathbb{Z}_p)$ acts on $\mathcal{A}_G(\mathbb{Z})$ (by composition). Actually $\operatorname{End}_{\operatorname{Rep} G(\mathbb{Q}_p)} \mathcal{A}_G(\mathbb{Z}_p)$ is the (unramified) Hecke algebra for $G(\mathbb{Q}_p)$. Therefore, $\mathcal{A}_G$ already has a bunch of operators built in.

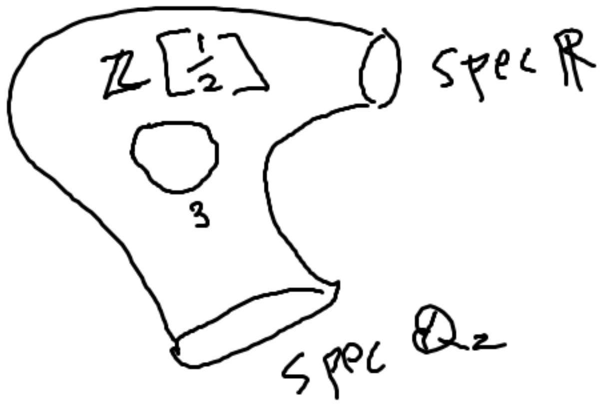


FIGURE 2.3. The prime ideal (3) of $\mathbb{Z}[\frac{1}{2}]$ is analogous to a knot/cycle in a 3-manifold

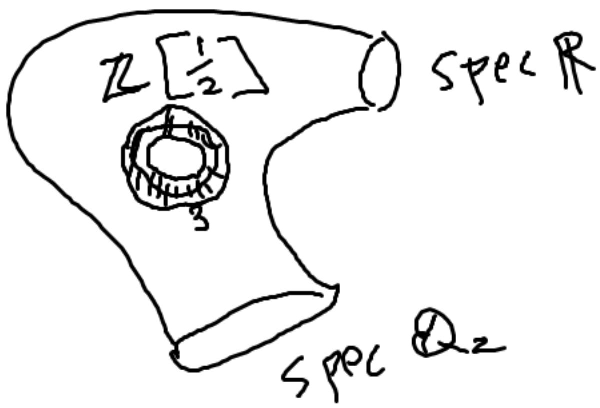


FIGURE 2.4. A solid tube is drawn around the knot corresponding to the prime ideal (3) of $\mathbb{Z}[\frac{1}{2}]$. This solid tube is removed to obtain $\mathbb{Z}[\frac{1}{6}]$.

Reciprocity. One of the central conjectures in the Langlands program is "reciprocity", which is a bijection

$$\{\text{eigenfunctions of Hecke operators}\} \leftrightarrow \{\text{homomorphisms } \pi_1(\text{the arithmetic object}) \to \check{G}\}.$$

$$\mathcal{A}_G \tilde{\leftrightarrow} \mathcal{B}_{\check{G}}$$

The equivalence $\mathcal{A}_G \tilde{\leftrightarrow} \mathcal{B}_{\check{G}}$ is of the aforementioned arithmetic TFQT $\mathcal{A}_G$ with a different arithmetic TFQT $\mathcal{B}_{\check{G}}$. The statement that these two arithmetic TQFT's are equivalent encodes the various layers of the Langlands program — one can put in a number field, local field, etc. and get this kind of bijection. The advantage of this formulation is not that one can obtain new statements (as one recovers already known statements when specializing to each arithmetic object) but rather that it gives a uniform way that one can talk about phenomena that occur throughout the Langlands program without having to separate various cases. one can recover many different aspects/statements in different parts of the Langlands program. The relative Langlands duality is a similar phenomenon — it exists in every layer of the Langlands program — and hence the above equivalence is a useful way to mentally organize and keep track of the relationships amongst the layers.

Exercise 1. **(a)** We need to verify

$$\sum_i q^{-\frac{i}{2}} \operatorname{tr} g|_{\mathbb{C}[Y]_i} = \frac{1}{(1-q^{-\frac{1}{2}}x)(1-q^{-\frac{1}{2}}x^{-1})}.$$

The right hand side is the product of two geometric series. The coefficient of $q^{-\frac{i}{2}}$ is $x^{-i} + x^{-i+2} + \cdots + x^i$, which is supposedly equal to $\operatorname{tr} x|_{\mathbb{C}[Y]_i}$.

$\mathbb{C}[Y] = \mathbb{C}[T^*\mathbb{A}^1] = \mathbb{C}[q,p]$. For $f \in \mathbb{C}[q,p]$, $x \cdot f(q,p) = f((q,p)\cdot x) = f(xq, x^{-1}p)$; in particular, $x \cdot q = xq$ and $x \cdot p = x^{-1}p$.

A basis of $\mathbb{C}[Y]_i$ is $\{q^i p^0, \ldots, q^0 p^i\}$, which, upon action by x, becomes $\{x^i q^i p^0, \ldots, x^{-i} q^0 p^i\}$. Therefore, $x|_{\mathbb{C}[Y]_i}$ is a diagonal matrix, with trace $x^i + x^{i-2} + \cdots + x^{-i}$.

(b) Equivalently, $x \in \mathbb{C}^*$ now diagonally acts on Y^2.

(c) $\operatorname{Ad}_g \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & x^2 b \\ x^{-2}c & d \end{bmatrix}$. Therefore, equivalently, $x^2 \in \mathbb{C}^*$ acts on $Y \times Y$, acting trivially on the first Y and in the manner of **(a)** on the second Y.

3. Day 1 Lecture 3 (Yiannis)

Today's first talk today was about (4d) TQFTs and the second talk was about Langlands as a TQFT. This third talk will be about **relative** Langlands as TQFTs. Here, relative = Langlands + some decorations.

Let's quickly recap what we have heard in the previous lectures. Everything is going to be 4d TQFTs, which means to a d-dimensional closed manifold C one attaches some state spaces $\mathcal{A}_G(C)$ which have categorical complexity opposite to the dimension, i.e. $\mathcal{A}_G(C)$ is a $4-d-1$-category, -1 = number, 0 = vector space, 1 = category.

Let G be a split connected reductive algebraic group over a local or global field. The followings are three typical examples of the **A side** theory of Langlands, where G gives rise to $\mathcal{A}_G(C)$ the A-side theory associated to C.

Example 3.1. C=global field F, such as a number field, or a function field $\mathbb{F}_q(\gamma)$ where γ is some smooth projective curve. Then C is some kind of 3 dimensional manifold.

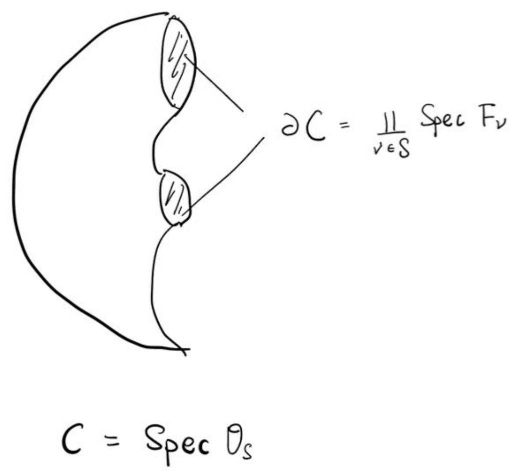
The corresponding $\mathcal{A}_G(C)$ would be some everywhere unramified automorphic functions $Fns(G(F)\backslash G(\mathbb{A}_F)/G(\hat{\mathcal{O}}))$ when F is a number field.

In the function field case there are two ways to think about it. One is the same as what we did in the number field case above. Another interpretaion would be the set of isomorphism classes of groupnoids of G-bundles of the curve γ over $\mathbb{F}_q$.

Example 3.2. Let F_v be some local field, for example $F_v = \mathbb{Q}_p$ or $F_q((t))$. They are of dimension 2. So we attach to it $\mathcal{A}_G(C)$ the category of $G(F_v)$-representations.

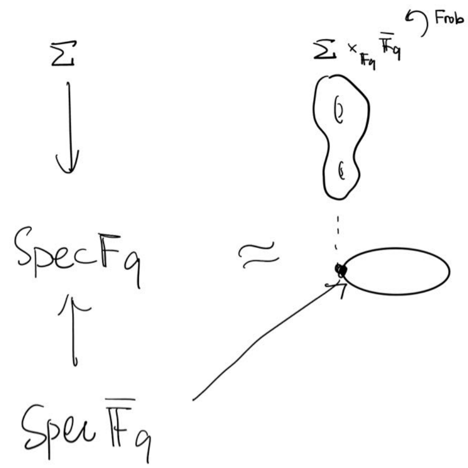
Example 3.3. Another example of dimension 2 is $\bar{\mathbb{F}}_q(\gamma)$. To this we attach the category of sheaves on Bun_G^γ, denoted by $\mathrm{Shv}(\mathrm{Bun}_G^\gamma)$.

Before we talk about the boundary conditions for the above examples, let's recall some examples of bordisms from previous lectures.



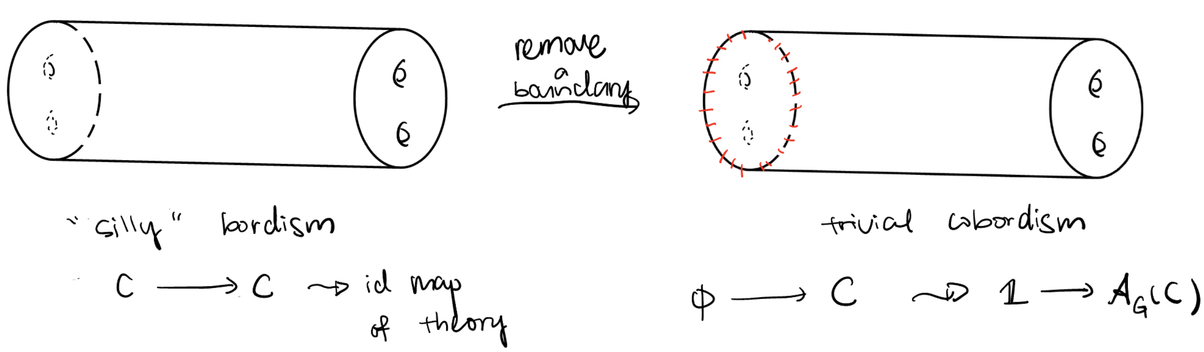
Let S be a finite set of places of the field F, we have $\mathcal{O}_S$, the ring of S-integers. Then $C = \mathrm{Spec}\, \mathcal{O}_S$ is a manifold with boundary, the boundaries being $\coprod_{v \in S} \mathrm{Spec}\, F_v$. We think of C as a bordism from empty set to the boundary ∂C, which, according to TQFT corresponds to a map from the unit object to the theory over the boundary $\mathcal{A}(\partial C) = \mathrm{Rep}\ G(F_v)$. Giving such map is the same as giving an object in $\mathcal{A}(\partial C)$, which we choose to be the automorphic forms possibly ramified at places in S, i.e. $\mathrm{Fns}(G(F) \backslash G(\mathbb{A}_F)/G(\hat{\mathcal{O}}^S))$.

The above example shows part of the relation between the theory of first example and the second by telling us how they glue together. There is also a relation between the first and the third given by some trace of Frobenius. Let me give the geometric picture behind this.



When given a smooth projective curve γ over $\mathbb{F}_q$, there is a map from γ to $\operatorname{Spec}\mathbb{F}_q$. Like David explained, $\operatorname{Spec}\mathbb{F}_q$ should be thought of as some kind of circles because the Galois group of its etale cover is essentially $\mathbb{Z}$. A point on the circle corresponds to choosing a algebraic closure (not canonical), i.e. a map from $\operatorname{Spec}\bar{\mathbb{F}}_q$ to $\operatorname{Spec}\mathbb{F}_q$. A fiber over the point is the curve γ base change to the chosen algebraic closure $\gamma_{\bar{\mathbb{F}}_q}$, which is a Riemann surface in the geometric analogue (if you replace the algebraic closure by $\mathbb{C}$). In a word, γ should be thought of as a family of Riemann surfaces over a circle. But this family is not topologically trivial. There is a Frobenius action and there is a construction of mapping cyclinder that David explained in the first lecture. The Frobenius action will tell you the monodromy going around the circle, which allows you to think of γ as something 3 dimensional and to take "trace" of the action. There is a categorical trace given by the Frobenius action that relates the theory of the first and the third examples.

So, what is relative Langlands and what is TQFT with boundary? Let me draw a picture that David didn't have time to draw. He would talk about it more tomorrow.



So the idea is that, in David's talk, our category is some manifold with bordisms. For each manifold you have a canonical but "silly" bordism which is from the manifold to itself (multiplying the manifold with the interval [0.1]). This is just the identity map. So the theory with boundary allows one to remove some boundaries and still think of it as a valid input for your period. What this means is that when enlarge the category, one can remove the boundary and think of the trivial cobordism (on the right hand side of the picture) as a morphism from the empty set to the manifold. Every manifold with boundary comes with a canonical morphism from empty set. Therefore, no matter what theory $\mathcal{A}_G(C)$ one have here (attached to the manifold C), will come with a canonical map from identity element. So the new theory (with boundary) is richer in the sense that it does not only give us some category but also an object of that category. But of course this "distinguished element" needs more data to produce and this data will be some space X with G-action in our theory. And this element will be denoted by $\mathcal{O}_{G,X}$.

I will want to talk about this additional data (for the "decorations") without talking about Hamiltonian and symplectic spaces. The prototype for a toy case is when G is a finite group. Let X be a vector space with G-action. In this case, (recall that $\mathcal{F}_G(C)$ is the field theory)

$$\mathcal{F}_G(C) = \mathrm{Map}(C, \mathrm{pt}/G) = \mathrm{Hom}(\pi_1(C), G)/\sim$$

where $\sim$ is G-conjugacy. In fact, $\mathcal{F}_G(C)$ has a lot of names such as Bun_G and Loc_G (since we are in the finite group case, they are the same). The A-side theory attached to C is $\mathcal{A}_G(C) = \mathrm{Fns}(\mathcal{F}_G(C))$. Therefore, the boundary condtion should be a distinguished function on $\mathcal{F}_G(C)$, which we shall denote by $\mathcal{O}_{G,X}$. The notation suggests that it is given by the extra data from G-space X.

Take a principle G bundle $\mathcal{E}_G \to C$. Since G acts on X, we could construct a vector bundle over C, denoted by $X \times^G \mathcal{E}_G$, by taking the product of X and $\mathcal{E}_G$ then modulo the diagnoal action.

Therefore, over $\mathrm{Bun}_G = \mathcal{F}_G(C)$, we can form a space

$$\mathrm{Bun}_G^X = \left\{(\mathcal{E}_G, \sigma) : \sigma = \text{section of } X \times^G \mathcal{E}_G\right\}$$

The distinguished function $\mathcal{O}_{G,X}$ that we associate to this data is defined by

$$\mathcal{E}_G \mapsto \#\{\sigma\} = \#(\mathrm{Bun}_G^X)_{\mathcal{E}_G} = \#X^\phi$$

The last equality comes from identifying $\mathcal{E}_G \in \mathrm{Map}(C, \mathrm{pt}/G)$ with a monodromy $\phi \in \mathrm{Hom}(\pi_1(C), G)/\sim$. Then $\#\{\sigma\}$ is nothing but the number of fixed points in X under the monodromy.

However in the Langlands setting, there is a difference between the above two description of $\mathcal{O}_{G,X}$, but the philosophy is the same.

The toy case is meant to give you an initial idea of how a space X gives rise to the boundary conditions. Actually X is not necessarily a vector space. It could be just a set.

Now back to our 3 examples, what are $\mathcal{O}_{G,X}$ in those cases? The table below is a summarization.

	example 1	example 2	example 3
C [dimension d manifolds]	F global field, $[d=3]$ e.g. number fields, function fields $\mathbb{F}_q(\gamma)$	F_ν local field, $[d=2]$ e.g. $\mathbb{Q}_p, \mathbb{F}_q((t))$.	$\bar{\mathbb{F}}_q(\gamma)$ $[d=2]$
$\mathcal{A}_G(C)$ $[(4-d-1)$-category$]$	$\mathrm{Fns}(G(F)\backslash G(\mathbb{A}_F)/G(\hat{\mathcal{O}}))$, (in function field case can be regarded as Bun_G^γ)	$G(F_v)$-representations	$\mathrm{Shv}(\mathrm{Bun}_G^\gamma)$
$\mathcal{O}_{G,X}$	period function $\mathcal{F}_X$	$\mathrm{Fns}(X(F_\nu))$	push forward the constant sheaf $\underline{\mathbb{C}} \in \mathrm{Shv}(\mathrm{Bun}_G^X)$ to $\mathrm{Shv}(\mathrm{Bun}_G)$
$\mathcal{B}_{\check{G}}(C)$	Fns("Langlands parameters going into $\check{G}$")	cohomological Langlands dual sheaves (local Langlands parameters)	$\mathrm{QC}^!(\mathrm{Loc}_{\check{G}}^\gamma)$
$\mathcal{O}_{\check{G},\check{X}}$	some "L-functions"	(will show up in future lectures)	(will show up in future lectures)

Let's first look at the third example. $\mathcal{O}_{G,X}$ is supposed to be some sheaf on the space $\mathrm{Bun}_G = \mathrm{Bun}_G^\gamma$. The extra data of X gives a space Bun_G^X over Bun_G. Pushing forward the constant sheaf gives one an element in $\mathrm{Shv}(\mathrm{Bun}_G)$.

Now it's time to see what the boundary conditions in other 2 examples are. First let's try to imagine what trace of Frobenius would give here. For the first case, we are supposed to give some automorphic functions. In the function field case it can be regarded as functions on the Bun_G over the curve γ. The functions of boundary condition in this case is going to be called period functions, which we denoted by $\mathcal{P}_X$, i.e. $\mathcal{O}_{G,X} = \mathcal{P}_X$. Here X is a variety over $\mathbb{F}_q$ with G-action. The function is on G-bundles so let's change our notation for G-bundles to be $[g]$ (which was $\mathcal{E}_G$ previously). The reason why we change to this notation is because we are identifying them with the adelic points in the double quotient. Now the function is defined by

$$\mathcal{P}_X([g]) = \#\left\{\text{sections of } [g] \times^G X \to \gamma\right\}$$

(this number is finite? can be rational?)

And if we denote the characteristic function of the $\hat{\mathcal{O}}$ points of X by $\mathbf{1}_X$, there is another way to write the period functions, which is the following:

$$\mathcal{P}_X([g]) = \sum_{\gamma \in X(F)} \mathbf{1}_X(\gamma g)$$

where F is the global field.

This construction of automorphic functions works for a more general class of functions. For a function ϕ on $X(\mathbb{A})$ with desired properties (say, rapidly decay, compactly supported, invariant by $X(\hat{\mathcal{O}})$ translation,), one can sum over left translation of ϕ by points in $X(F)$. Because when X is a affine variety, $X(F)$ is a discrete set, the sum makes sense. The resulting function is then invariant under left $G(F)$ action since the set $X(F)$ is invariant under $G(F)$. This kind of construction is called theta-series in general. In specific context it might also be called Poincare series, (pseudo-)Eisenstein series, etc.

Finally, what is the boundary condition we want to give for the second example? What we are going to put here is some $G(F_v)$-representations. So it is just the space of functions on $X(F_v)$. It is a $G(F_v)$-representation because G acts on X.

Now, let me take some time to explain how $\mathrm{Bun}_G^\gamma(\mathbb{F}_q)$ is identified with the adelic points $G(\mathbb{F}_q)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}})$.

Given a G-bundle $\mathcal{E}$, one can find a open subset U of the curve γ such that $\mathcal{E}$ restricting to U is isomorphic to the trivial G-bundle. The isomorphism is of course not given, but one can choose such an isomorphism. The complement of U in γ is a set of finite points. So to get $\mathcal{E}$ we just neet to glue the G-bundle on U with the G-bundles on formal disks around each point not in U. The details of this identification are left as exercise.

Now everything we talked about above is only one-side of Langlands, which is usually called the A-side of the theory. But Langlands has two sides and the duality between both sides. So what is the B-side? B-side should be the theory associated to the dual group $\check{G}$.

Recall that in the first example, the A-side of the theory is just functions $Fns(Bun_G(F))$. The B-side theory $B_{\check{G}}(C)$ is also functions but it's, roughly speaking, functions on the Langlands parameters into $\check{G}$. And what is the duality here? The duality is an isomorphism between the two spaces of functions, which is some noncommutative version of Fourier transform. For automorphic functions, you are supposed to do spectral decomposition into eigenfunctions of the Hecke operators. Nothing will be meaningful without Hecke operators. Implicitly behind these equivalences we have Hecke operators and these operators allow you to spectrally decompose the spaces and on the B-side these are supposed to be joint eigenvalues of the operators. The most nontrivial part is that originally there is no Galois theory on the A-side but the Hecke operators relates it to the B-side with Langlands parameters which related to the Galois representations.

In the third example where A-side theory is some sheaves on $\mathrm{B}un_G$, the B-side theory is some quasi-coherent sheaves $QC^!(Loc_{\check{G}}^\gamma)$. I am not going to say anything about this but just mentioning it for completion.

The B-side theory in the second example, corresponding to the A-side (which are representations of $G(F_v)$) should also be some kind of spectral decomposition. The best known version of this categorical Langlands dual is the one proved by Fargues and Scholze. There is another version proved by Xinwen Zhu.

The relative Langlands duality is supposed to take care of the third column–the boundary conditions. There should be a corresponding boundary conditions $\mathcal{O}_{\check{G},\check{X}}$ for the B-theory.

For instance, in the first example, where the boundary condition is some period functions or thera theories, the corresponding B-side boundary condition is some "L-function". All the evidenve we have for this duality comes either from this case, where we have some knowledge about periods and L-functions, or from the second case. But let me not say anything about that.

I actually lied a bit about this boundary condtions given by X or $\check{X}$. In fact they are in general associated to the analogue of their cotangent spaces. So the conjectural relative Langlangs duality is actually between (G, M) and $(\check{G}, \check{M})$ where $M/\check{M}$ are Hamiltonian $G/\check{G}$-space. And in some example, M has the interpretation as a cotangent space T^*X.

Finally let me give a number theoretic example of identification of theta series with periods, which is the example from Tate thesis. Let's consider $X = \mathbb{A}^1$, with an action of $G = G_m$. I am going to take this function

$$\Phi = (\otimes_{\nu<\infty} \mathbf{1}_{\mathbb{Z}_p}) \otimes \Phi_\infty, \Phi_\infty \in \mathcal{S}(\mathbb{R})$$

where we can take $\Phi_\infty = e^{-x^2}$, the Guassian. The theta series we construct from this Φ is

$$\sum_{q \in X(\mathbb{Q})} \Phi(qg), g = (1, \cdots, 1, x) \in [G]$$

Now, in order for a rational number to be in $\mathbb{Z}_p$ for every p, it has to be an integer. So we have

$$\sum_{q \in X(\mathbb{Q})} \Phi(qg) = \sum_{n \in \mathbb{Z}} \Phi_\infty(nx) = \sum_{n \in \mathbb{Z}} e^{-n^2 x^2}$$

4. Day 2 Lecture 1

4.1. **More on TQFTs.**

Last time (Day 1 Lecture 1), we talked about the notion of a TQFT as a means to linearize maps into $BG = \bullet/G$. In Day 1 Lecture 3, we considered the example of $|G| < \infty$, a TQFT $\mathcal{A}_G$ which comes with the data of fields $\mathcal{F}_G$ so that

$$\mathcal{F}_G(C) = \mathrm{Bun}_G(C) = \mathrm{Maps}(C, \bullet/G) = \mathrm{Hom}(\pi_1(C), G)/G.$$

Note that under the assumption that G is finite, we have that $\mathrm{Loc}_G(\Sigma) = \mathrm{Bun}_G(\Sigma) = \mathrm{Maps}(\Sigma, \bullet/G)$, and we will freely assume this in this subsection. For example, we have that

$$\mathcal{F}_G(S^1) = \mathrm{Maps}(S^1, \bullet/G) = \mathrm{Hom}(\mathbb{Z}, G)/G = G/G,$$

where the quotient is taken with respect to the conjugation action.

We can also replace $\bullet/G$ with a finite groupoid (stack) $\mathfrak{X} = X/G$, where X is a G-space, and take $\mathcal{F}_G$ such that

$$\mathcal{F}_G(S^1) = \mathrm{Loc}_G^X(S^1) = \mathrm{Bun}_G^X(S^1) = \mathrm{Maps}(S^1, X/G) = \{x \in X,\ g \in G,\ g \cdot x = x\}/G$$

(and similarly for general Σ instead of S^1).

Observe that we have a map $\mathrm{Loc}_G^X(\Sigma) \to \mathrm{Loc}_G(\Sigma)$ induced by the map $X/G \to \bullet/G$, since $\mathrm{Loc}_G(\Sigma) = \mathrm{Maps}(\Sigma, \bullet/G)(= \mathrm{Bun}_G(\Sigma))$.

Once we fix such a G-space X, we have a distinguished element

$$\Theta_{G,X}(\Sigma) \in Z_G^3(\Sigma) = \mathbb{C}[\mathrm{Loc}_G(\Sigma)]$$

defining a boundary theory $\Theta_{G,X}$.

The 4d (arithmetic) TQFT $\mathcal{Z} = \mathcal{A}_G$ can be obtained by taking a field $\mathcal{F}_G = \mathrm{Bun}_G(-) = \mathrm{Maps}(-, \bullet/G)$ followed by linearization (taking functions/sheaves on the spaces), as abstracted in the following diagram:

$$\begin{tikzcd}
\begin{array}{c}\text{Manifolds/}\\ \text{Bordism}\end{array} \arrow[rr, "\mathcal{A}_G"] \arrow[dr, "\mathcal{F}_G"'] & & \begin{array}{c}\text{Linear Categories/}\\ \text{Morphisms}\end{array} \\
& \begin{array}{c}\text{Groupoids (Stacks)/}\\ \text{Correspondences}\end{array} \arrow[ur, "\text{Functions/Sheaves}"'] &
\end{tikzcd}$$

Let us discuss the functoriality of $\mathcal{A}_G$. We know that $\mathcal{A}_G$ factors via $\mathcal{F}_G$, so we begin by describing functoriality of $\mathcal{F}_G$. Given a bordism

$$M \colon \partial_{\text{in}} \to \partial_{\text{out}},$$

we have a correspondence via the following diagram

$$\begin{tikzcd}
& \mathcal{F}_G(M) = \mathrm{Bun}_G(M) \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathcal{F}_G(\partial_{\text{in}}) = \mathrm{Bun}_G(\partial_{\text{in}}) & & \mathcal{F}_G(\partial_{\text{out}}) = \mathrm{Bun}_G(\partial_{\text{out}}).
\end{tikzcd}$$

Then we get a morphism $\mathcal{A}_G(M) \colon \mathcal{A}_G(\partial_{\text{in}}) \to \mathcal{A}_G(\partial_{\text{out}})$ by taking the pullback along the first morphism f and pushing forward along the second one g.
Moreover, when we are given a G-space X, we can define a distinguished object $\Theta_{G,X}(\Sigma) \in \mathcal{A}_G(\Sigma)$. We obtain this by looking at an extended bordism $M \colon \varnothing \to \Sigma$,

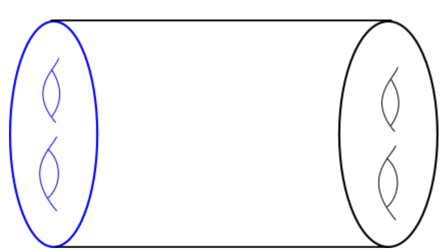


FIGURE 4.1. $M \colon \varnothing \to \Sigma$

which gives us a correspondence

$$\begin{tikzcd}[column sep=small] & \mathrm{Bun}_G^X(\Sigma) = \mathrm{Maps}(\Sigma, X/G) \arrow[dl] \arrow[dr] & \\ \mathcal{F}_G(\varnothing) = \bullet & & \mathcal{F}_G(\Sigma) = \mathrm{Maps}(\Sigma, \bullet/G). \end{tikzcd}$$

which in turn gives a morphism $\mathbb{1} \to \mathcal{A}_G(\Sigma)$, or equivalently, an object $\Theta_{G,X}(\Sigma)$ in $\mathcal{A}_G(\Sigma)$. This is how we'll get period functions/sheaves (and similarly L-functions/sheaves on the B-side (Spectral side).)

4.2. **The Automorphic Theory.**

Now we consider the case where G is a reductive group over a field $\Bbbk$ to get a 4d TQFT $\mathcal{A}_G$ which extends to arithmetic manifolds. $\mathcal{A}_G$ consists of the topology of moduli spaces of algebraic G-bundles.

In the global geometric setting: Σ is an algebraic curve over $F = \overline{F}$ (e.g. $\mathbb{C}$, $\overline{\mathbb{F}_q}$) (2-dimensional) and we get that $\mathcal{F}_{\mathcal{A}_G}(\Sigma) = \mathrm{Bun}_G(\Sigma)$.

Example 4.1. $\mathcal{F}_{\mathbb{G}_\mathrm{m}}(\Sigma) = \mathrm{Pic}(\Sigma)$.

Example 4.2. If $\Sigma = D_F^\times = \mathrm{Spec}\,(F((t)))$, then we have

$$\mathcal{F}_G(D_F^\times) = \mathrm{Bun}_G(D_F^\times) = \bullet/G(F((t))).$$

Example 4.3. If Ξ is a curve over $\mathbb{F}_q$, then we have

$$\mathcal{F}_G(\Xi) = \mathrm{Bun}_G(\Sigma)^{\mathrm{Fr}} = \mathrm{Bun}_G(\Sigma)(\mathbb{F}_q).$$

(Here Σ is the base change to $\overline{\mathbb{F}_q}$ of Ξ, and $\mathrm{Bun}_G(\Sigma)^{\mathrm{Fr}}$ is the space of fixed points with respect to the Frobenius.)

Example 4.4. In particular, if $\Sigma = D_{\mathbb{F}_q}^\times = \mathrm{Spec}\,(\mathbb{F}_q((t)))$, then we have

$$\mathcal{F}_G(D_{\mathbb{F}_q}^\times) = \mathrm{Bun}_G\left(D_{\overline{\mathbb{F}_q}}^\times\right)^{\mathrm{Fr}} = G(\mathbb{F}_q((t)))/^{\mathrm{Fr}}G(\mathbb{F}_q((t))) \supset \bullet/G(\mathbb{F}_q((t))).$$

Now we want to linearize the above spaces. Note that $\mathcal{A}_G$ sends 3-manifolds to vector spaces and 2-manifolds to linear categories.

Example 4.5. For a 3-manifold Ξ in Example 4.3, we have

$$\mathcal{A}_G(\Xi) = \mathrm{Fun}(\mathrm{Bun}_G(\Sigma)(\mathbb{F}_q)),$$

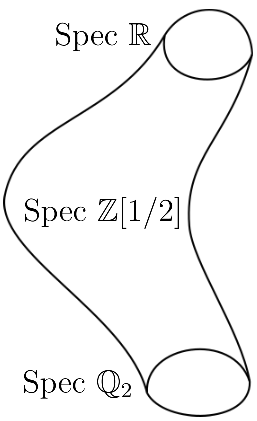
which is the vector space of unramified automorphic forms.

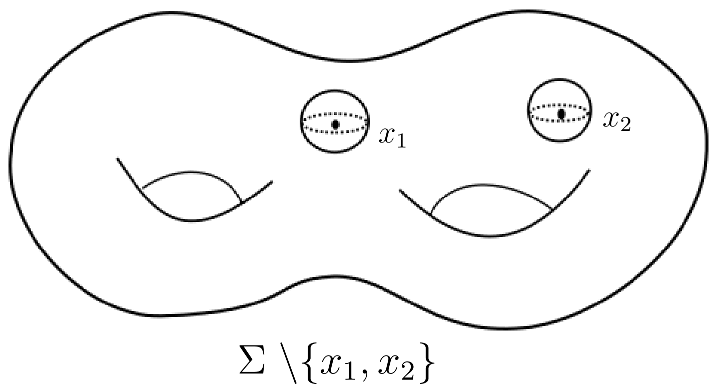
Example 4.6. For a 2-manifold $D_{\mathbb{F}_q}^\times$ in Example 4.4, we have

$$\mathcal{A}_G(D_{\mathbb{F}_q}^\times) = \mathrm{Shv}(\mathrm{Bun}_G(D_{\mathbb{F}_q}^\times)) = \mathrm{Shv}(G(\mathbb{F}_q((t)))/^{\mathrm{Fr}}G(\mathbb{F}_q((t)))),$$

the automorphic category attached to the punctured disk and contains $\mathrm{Shv}(\bullet/G(\mathbb{F}_q((t)))) = \mathrm{Rep}^{\mathrm{smooth}}(G(\mathbb{F}_q((t))))$, the category of smooth representations of the group over the local field $\mathbb{F}_q((t))$, which is what is studied in classical local Langlands.

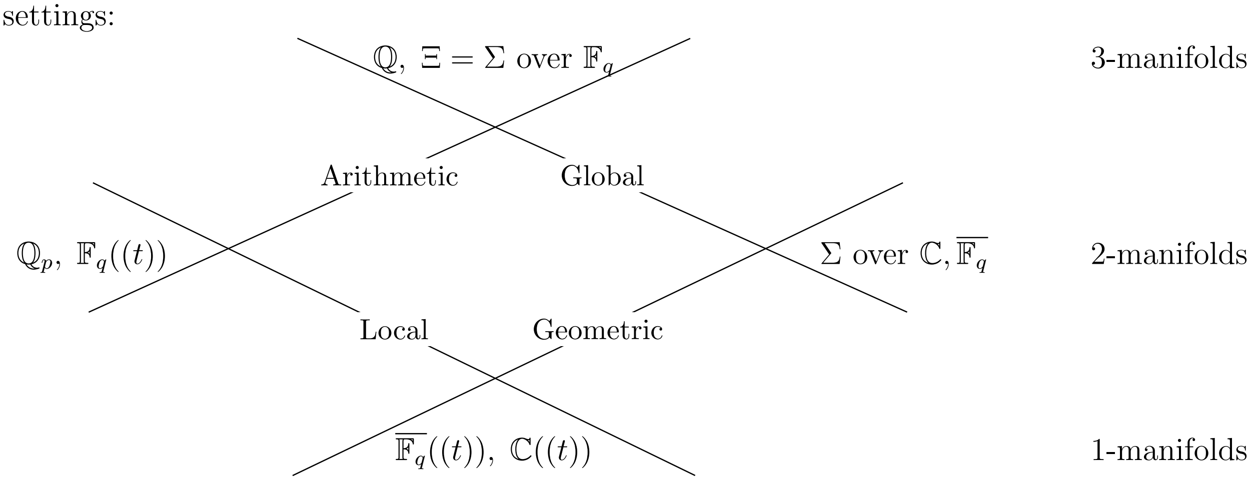
Example 4.7. Now, one might think that the TQFT only encodes information about the unramified parts, but in fact, it also sees the ramified parts:





We had Spec $(\mathbb{Z}[1/2])$, thought of as a 3-manifold with 2 boundary components Spec $(\mathbb{Q}_2)$ and Spec $(\mathbb{R})$. We can similarly take a curve Σ over a finite field $\mathbb{F}_q$, remove a couple of points x_1, x_2, and then look at $\mathrm{Fun}(\mathrm{Bun}_G(\Xi|x_1, x_2))$ (functions on Bun_G with some extra data associated to the two points). This gives an object in Rep $G(\mathbb{F}_q((t)))^2$. We omit the details here, but the point is that the TQFT certainly knows what to do about ramifications: we just remove a few things so that rather than getting a vector space to a closed 3-manifold, we get an object in the category associated to the boundary, which in this case is Rep $G(\mathbb{F}_q((t)))^2$.

The following diamond shows the big overview of the various global/local arithmetic/geometric settings:



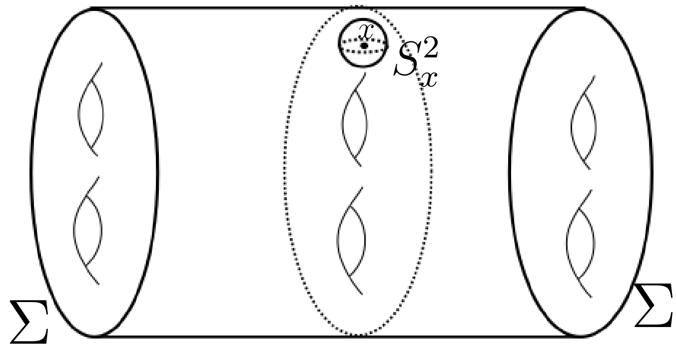


FIGURE 4.2. The bordism $(\Sigma \times I) \setminus D_x^3 : S_x^2 \amalg \Sigma \to \Sigma$

We haven't made precise what sheaf theory we're working with. For example, for global geometric Langlands, we have

$$\mathcal{A}_G(\Sigma) = \mathrm{Shv}^?_{\mathcal{N}}(\mathrm{Bun}_G(\Sigma)), \text{ where } ? = \begin{cases} \text{dR (de Rham)}, & D\text{-modules}, \\ \text{B (Betti)}, & \text{sheaves of vector spaces}, \\ \text{ét (étale)}, & \text{étale constructible sheaves.} \end{cases}$$

(To be precise, there is a technical modification necessary to make things work: we need to take sheaves with nilpotent support. For example, if $G = \mathbb{G}_m$, then we have $\mathrm{Bun}_G(-) = \mathrm{Pic}(-)$, and the adjective "nilpotent support" lets us work with locally constant sheaves on Pic instead of all sheaves.)

4.3. **Symmetries of Observables, Hecke Operators.**

Let us go back to the finite group setting. Let H be a finite group, so that $\mathrm{Bun}_H(-) = \mathrm{Loc}_H(-)$. We look at the 4d TQFT $\mathcal{Z}_H^4$ associated to the finite group H that we talked about yesterday, so that $\mathcal{Z}_H^4(\Sigma) = \mathrm{Vect}(\mathrm{Bun}_H(\Sigma))$. This category has nice symmetries, and is acted on by the observables, i.e. by $\mathcal{Z}_H^4(S^2)$ (a.k.a the Hecke category/category of line operators), as explained in Lecture 1 yesterday. The action can be obtained by hitting the bordism in Figure 4.2 with $\mathcal{Z}_H^4$ (after choosing $x \in \Sigma$):

Explicitly,

$$\mathcal{Z}_H^4(S_x^2) = \mathrm{Vect}(\bullet/H) = (\mathrm{Rep}(H), \otimes) \ \ (\text{since } \mathrm{Bun}_H(S^2) = \mathrm{Loc}_H(S^2) = \bullet/H),$$

acts on $\mathcal{Z}_H^4(\Sigma)$ as follows:

- For each $x \in \Sigma$, we have the map
$$\mathrm{Bun}_H(\Sigma) \xrightarrow{(-)|_x} \mathrm{Bun}_H(x) = \bullet/H$$
- Given $V \in \mathcal{Z}_H^4(S^2) = \mathrm{Rep}(H)$, we can pull it back along the above map to get a vector bundle $\mathcal{V}_x$ on $\mathrm{Bun}_H(\Sigma)$.

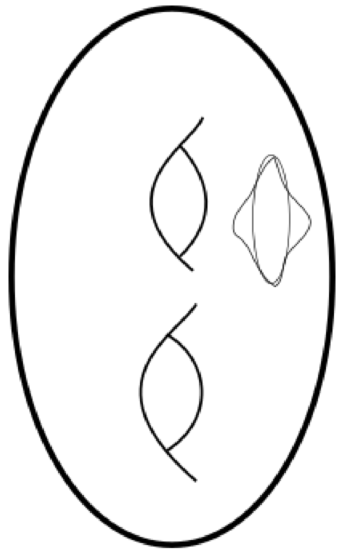


FIGURE 4.3. $\Sigma \amalg_{\Sigma \setminus \{x\}} \Sigma = \Sigma \amalg_{D_x} (\;) = \Sigma \amalg_{D_x} (D_x \amalg_{D_x^\times} D_x) \colon \Sigma \to \Sigma$

- Then V acts on $\mathcal{E} \in \mathcal{Z}^4_H(\Sigma)$ by taking

$$V \cdot \mathcal{E} := \mathcal{V}_x \otimes \mathcal{E}.$$

The operators that we will see on the B-side (Spectral side) of the Langlands correspondence will be exactly like this.

Now we come back to the case where G is a reductive group. Taking

$$\mathcal{Z}(-) = \mathcal{A}_G(-) = \mathrm{Shv}^?_N(\mathrm{Bun}_G(-)),$$

(or we can take the product with a circle and take trace of the Frobenius; for a curve over a finite field, we replace sheaves with automorphic functions.)

To get the Hecke operators, we again take $x \in \Sigma$, and look at the bordism (Figure 4.2). This should give a correspondence after taking the space of fields and then we can linearize to get the Hecke action. But of course, we can't just do this due to technical difficulties: $I = [0, 1]$ is not an algebraic curve, so we have to make sense of how to parse this in algebraic geometry. Since everything is supposed to be locally constant, we can imagine shrinking this picture, to get something like Σ glued to a raviolo along a disk around x (see Figure 4.3 below).

The raviolo should be thought of as $D \amalg_{D^\times} D$ and is the algebro-geometric object playing the role of the 2-sphere. This gives us a correspondence after taking the space of fields, i.e., the Hecke correspondence:

$$\begin{tikzcd}
& \mathrm{Bun}_G(\Sigma \amalg_{\Sigma\setminus\{x\}} \Sigma) = \mathcal{H}ecke_x \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G(\Sigma) & & \mathrm{Bun}_G(\Sigma).
\end{tikzcd}$$

Here an element of $\mathrm{Bun}_G(\Sigma \amalg_{\Sigma\setminus\{x\}} \Sigma)$ is nothing but a pair of G-bundles identified away from the point x, but can also be thought of as a single G-bundle on Σ along with some local data associated to the local modification at x. Now we can take automorphic functions/sheaves and pullback along the first map, multiply/tensor by something in $\mathcal{A}_G()$ and pushforward along the second map to get an operator/functor. This gives us the Hecke action of $\mathcal{A}_G()$ on $\mathcal{A}_G(\Sigma)$.

Example 4.8.

- (Arithmetic setting, from Day 1 Lecture 2) Recall that in the arithmetic context, we get the action of the Hecke operators by taking the object associated with the number field, removing a prime (i.e., inverting the prime) and gluing it back in, and now we have endomorphisms of the thing that we glued back in. The Hecke action in that setting can be thought of as coming exactly from the bordism above.

- If $G = \mathbb{G}_{\mathrm{m}}$, then $\mathrm{Bun}_G(= D_x \amalg_{D_x^\times} D_x) = \mathbb{C}((t))^\times/\mathbb{C}[[t]]^\times \cong \mathbb{Z}$, and the Hecke action of $\mathcal{A}_G(D_x \amalg_{D_x^\times} D_x) = \mathrm{Shv}(\mathbb{Z})$ on $\mathcal{A}_G(\Sigma) = \mathrm{Shv}(\mathrm{Pic}(\Sigma))$ is essentially induced by the action of $\mathbb{Z}$ on $\mathrm{Pic}(\Sigma)$ defined as $n\colon \mathcal{L} \mapsto \mathcal{L}(nx)$.

5. Day 2 Lecture 2

5.1. **A Miracle.** We start with Riemann-zeta functions, including all $\mathbb{Z}$. In some books, a different variable $y = x^2$ is used in the following formulas.

Riemann showed that

$$\int_0^\infty x^s \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x^2} d^x x = 2\pi^{-s/2}\Gamma(s/2)\zeta(s) = L(\chi, 0) \tag{5.1}$$

where

$$\chi : x \mapsto |x|^s, \tag{5.2}$$

$$\zeta(s) = \prod_p \frac{1}{1-p^s}, \tag{5.3}$$

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x^2} = \Theta(x), \tag{5.4}$$

the Jacobi theta function, and we denote

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2). \tag{5.5}$$

Also, just as a notation $d^x x = \frac{dx}{x}$.

The above is the Mellin transform, understood as the multiplicative Fourier transform, of the the theta series.

We note that for the theta function, we have, under the Poisson summation formula

$$\theta(x) \xleftrightarrow{PSF} x^{-1}\theta(x^{-1}) \tag{5.6}$$

and similarly

$$\zeta_\infty(s) \longleftrightarrow \zeta(1-s). \tag{5.7}$$

We note this is a case of the period function, denoted P_X for $X = \mathbb{A}^1$ with a $G = \mathbb{G}_m$ action.

We note that there's no general theory of writting L functions in periods.

In algebraic geoemetry, we have

$$\text{"Schur Variety"} \overset{\text{period}}{\cong} \text{"Schur variety"}. \tag{5.8}$$

For Adelics, (5.1) is generalized as

$$\int_{[G_m]} |x|^s \sum_{\gamma \in F = \mathbb{A}^1(F)} \Phi(\gamma x) d^x x \tag{5.9}$$

Here, it is an integral over

$$[G_m] = G(F)\backslash G(\mathbb{A}) \quad or \quad G(F)\backslash G(\mathbb{A})/G(O). \tag{5.10}$$

$$P_X(x) \sum_{\gamma \in F = \mathbb{A}^1(F)} \Phi(\gamma x) \in S(X(\mathbb{A})) \tag{5.11}$$

is the period, where $S(X(\mathbb{A}))$ denotes the Schwartz functions, and X is local, and

$$\Phi(\gamma x) = \prod_{p<\infty} 1_{\mathbb{Z}_p} \otimes e^{-\pi x_\infty^2}, \tag{5.12}$$

with a $G(\hat{\mathbb{Z}})$ unit action on $1_{\mathbb{Z}_p}$.

Here we use the idele class character

$$F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times. \tag{5.13}$$

We give the following remarks

1. The above two differ in that are the automorphic forms vs automorphic functions.
2. We may consider the above to be a TQFT with two boundaries, one with $G(\mathbb{R})$ action and one with $G(\mathbb{Q}_p)$ action.

(3) For X to be smooth affine over O_v, $1_{X(O_v)}$ is a basic function.
(4) This integral is a special case of the Construction

$$\langle \text{Automorphic Forms}, \text{Periods} \rangle = L\text{- functions} \tag{5.14}$$

(5) 5.6 can be seen as switching the two subspaces of $T^*\mathbb{A}^1 = M$.That is, under PSF,

$$\mathbb{A}^1 \longleftrightarrow O(\mathbb{A}^1)^* \tag{5.15}$$

is invariant in $T^*\mathbb{A}^1$. Here M is a Hamiltonian G space.

Now we may answer the question *What makes Riemann's construction click?*

Unfolding the Tate integral (ignore $\gamma = 0$)

$$\int_{\mathbb{A}^1} |x|^s \Phi(\gamma x) d^x x = \prod_{p \le \infty} \int_{\mathbb{Q}_p^\times} \Phi_p(x) d^x x \tag{5.16}$$

Fix $p < \infty$, we have

$$\int_{\mathbb{Z} = \mathbb{Q}_p^\times / \mathbb{Z}_p^\times} |x|^s 1_{\mathbb{Z}_p^\times}(x) d^x x = \sum_{i \ge 0} p^{-is} = \frac{1}{1 - p^s} = \zeta_p(s) \tag{5.17}$$

This is a MIRACLE!!! Also, here, for $i = 0$, we get the sum to be 1.

5.2. **Mellin transform of the basic function.** Let $N := \mathcal{S}(\mathbb{Q}_p)^{\mathbb{Z}_p^\times}$. It is a module of the Hecke algebra

$$\mathcal{H}(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times) = \mathbb{C}[\mathbb{Q}_p^\times / \mathbb{Z}_p^\times] = \mathbb{C}[\mathbb{Z}]$$

Let $1_{\mathbb{Z}_p} \in N$ be the constant function on $\mathbb{Z}_p$.

Does $\zeta_p(s)$ appear in the module structure of N? The answer is no. N is a free $\mathbb{Z}$ module of rank 1 generated by $1_{\mathbb{Z}_p}$, but consider

$$N^* := S(\mathbb{Q}_p^\times)^{\mathbb{Z}_p^\times} = \mathbb{C}[\mathbb{Z}]$$

which contains the element $1_{\mathbb{Z}_p^\times}$ and also has an action of $\mathcal{H}(\mathbb{Q}_p^\times, \mathbb{Z}_p^\times)$ action, the Mellin transform of $1_{\mathbb{Z}_p^\times}$ is 1 but not $\zeta_p(s)$.

ζ starts to appear only when we think of the module as an inner product space. The inner product $\langle , \rangle$ comes from the embedding $N \hookrightarrow L^2(\mathbb{Q}_p)$ which is the same as $L^2(\mathbb{Q}_p^\times)$.

To clarify a pedantic point, when doing integration, we are really considering $1_{\mathbb{Z}_p}(x)|dx|^{1/2}$ where $|dx|^{1/2}$ is called half density.

5.3. **Plancherel Formula.** We want a formula of the form

$$\langle f_1, f_2 \rangle = \langle \hat{f}_1, \hat{f}_2 \rangle \tag{5.18}$$

where $\hat{f}$ is the Mellin transform of f. The functions live on $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \equiv \mathbb{Z}$, so their Mellin transform live on the dual of $\mathbb{Z}$ which is $S^1 \subset \mathbb{C}$, i.e. unitary characters.

It can be computed that

$$\widehat{1_{\mathbb{Z}_p}|dx|^{1/2}}(z) = \frac{1}{1 - p^{-1/2}z}|d^\times z|^{1/2}$$

here $z \in S^1$ corresponds to p^{-s} before and the shift of $p^{-1/2}$ is due to the half density $|dx|^{1/2}$.

The formula (5.18) in this case reads

$$\langle 1_{\mathbb{Z}_p}|dx|^{1/2}, 1_{\mathbb{Z}_p}|dx|^{1/2} \rangle = \int_{S^1} \frac{1}{(1 - p^{-1/2}z)(1 - p^{-1/2}z^{-1})} d^\times z$$

And we can also do this with action of Hecke algebra. Let $h \in \mathcal{H} = \mathbb{C}[\mathbb{Z}]$, taking h action on the first term will multiply the integrand by $\hat{h}$ on the right. Namely

$$\begin{aligned} \langle h \cdot 1_{\mathbb{Z}_p}|dx|^{1/2}, 1_{\mathbb{Z}_p}|dx|^{1/2} \rangle &= \int_{S^1} \frac{\hat{h}(z)}{(1 - p^{-1/2}z)(1 - p^{-1/2}z^{-1})} d^\times z \\ &= \int_{S^1} \hat{h}(z)\zeta_p(\frac{1}{2} + s)\zeta_p(\frac{1}{2} - s) d^\times z \end{aligned}$$

Hidden here is the dual $\mathbb{G}_m$ which is the Langlands dual of the original $\mathbb{G}_m$ and the dual $T^*\mathbb{A}^1 =: \check{M}$. When we setup general formalism for L-functions we are going to say that this zeta function here is associated to the action of this dual $\mathbb{G}_m$ action on the same space by coincidence. But this is just a very special case of the duality.

5.4. **Satake isomorphism.** Let us now recall the classical Satake isomorphism.

Let $\mathcal{H}_G = \mathcal{H}(G(F), G(O))$ be the spherical Hecke algebra of a reductive, connecte, split group scheme G over F.

Then the classical Satake isomorphism states that $\mathcal{H}_G \simeq \mathcal{H}_T^W$ where $T = B/N$. This is also equivalent to $\mathcal{H}_G \simeq \mathbb{C}[\mathrm{Rep}\ \check{G}]$ due to the following family of isomorphisms:

$$\mathcal{H}_G \simeq \mathcal{H}_T^W = \mathbb{C}[X_\bullet(T)]^W = \mathbb{C}[\check{T}]^W = \mathbb{C}[\check{G}]^{\check{G}} \simeq \mathbb{C}[\mathrm{Rep}\ \check{G}]. \tag{5.19}$$

The latter isomorphism $\mathbb{C}[\check{G}]^{\check{G}} \simeq \mathbb{C}[\mathrm{Rep}\ \check{G}]$ is given by the character map from the Grothendieck group of the category Rep $\check{G}$ to the ring of invariants of the adjoint action of $\check{G}$ on itself.

The construction of the Hecke isomorphism will not be reviewed here, we only would like to note that it uses the $\mathcal{H}_G \otimes \mathcal{H}_T$-bimodule $S(N(F)\backslash G(F))^{G(O)}$.

Also one can say that this whole picture only becomes completely justified by the geometric Satake, which turns $\mathcal{H}_G$ into a semi-simple category and states that it is equivalent to Rep $\check{G}$.

Moreover, the classical Satake isomorphism does not work very well from the point of view of analysis. To see that, let us consider the inner products on both sides.

On the right-hand side we have a canonical basis $\{s_\lambda\}$ of the Weyl characters of the irreducible representations $\{V_\lambda\}$ with **lowest** weight λ, i.e. λ's are the antidominant cocharacters of T (the choice to use the lowest weights $\lambda \in X_\bullet^-(T)$ is necessary to make the formulas below work).

At the same time, on the left-hand side there is an inner product coming from the embedding $\mathcal{H}_G \hookrightarrow L^2(G)$.

Naturally, the question arises: does the canonical basis on the right-hand side correspond to an orthonormal basis on the left-hand side? The answer is no.

Denote by $\hat{h}$ the image of $h \in \mathcal{H}_G$ under the Satake isomorphism and define h_λ to be the elements such that $\hat{h}_\lambda = s_\lambda$.

There is of course an orthogonal basis on the left-hand side which is naturally parametrized by the same set $\{\lambda \in X_\bullet^-(T)\}$: one just needs to take characteristic functions $\mathbb{1}_{g(O)\omega^\lambda G(O)}$ of double $G(O)$-cosets. However, it is not the basis we are looking for: $h_\lambda \neq \mathbb{1}_{G(O)\omega^\lambda G(O)}$ and moreover $\{h_\lambda\}$ are not orthogonal with respect to the inner product coming from $\mathcal{H}_G \hookrightarrow L^2(G)$.

Hence we arrive at the following question: what is the natural inner product on $\mathcal{H}_G$ that makes h_λ's orthogonal?

To answer it let us first recall what we mean by orthogonality of s_λ's: we can think of $\{s_\lambda\}$ as of characters of the maximal compact subgroup $\check{G}_c \subset G$ and then one have $\int\limits_{\check{G}_c} s_\lambda \overline{s_\mu} = \delta_{\lambda\mu}$. Using Weyl integration formula this can be rewritten as

$$\frac{1}{|W|}\int\limits_{\check{T}_c} s_\lambda(t)\overline{s_\mu(t)}d_{Weyl}(t) = \int\limits_{\check{G}_c} s_\lambda \overline{s_\mu} = \delta_{\lambda\mu} \tag{5.20}$$

where $d_{Weyl}(t) = \prod\limits_{\check{\alpha} \in R(\check{G})} (1 - e^\lambda)$ is the Weyl measure from the Weyl character formula (here $e^\lambda : T \to \mathbb{C}^\times$ is the map given by $\lambda \in X^\bullet(T)$).

We will use this Weyl measure to build an inner product on $\mathcal{G}$ which makes h_λ orthogonal. The previous discussion shows that we need to introduce some non-trivial modification to it to make it work. It turns our that the right modification is provided by the Macdonald formula over the group G: we can define

$$\langle h, \mathbb{1}_{G(O)}\rangle = \langle h * \mathbb{1}_{G(O)}, \mathbb{1}_{G(O)}\rangle = \frac{1}{|W|}\int\limits_{T_{\check{G}}} \frac{\hat{h}(t)}{\prod\limits_{\check{\alpha}\in R(\check{G})}(1 - q^{-\frac{1}{2}}e^{\check{\alpha}})} d_{Weyl}(t) \tag{5.21}$$

The modified measure in the right-hand side is the Plancherel measure for the group.

The modification factor can be rewritten on terms of the (q-character of) adjoint representation of $\check{G}$ on it's Lie algebra. The conceptual explanation for this is provided by the derived Satake isomorphism.

6. Day 2 Lecture 3

Let F be either the local fields $\mathbb{Q}_p, \mathbb{F}_p((t))$, which are 2-dimensional, or the geometric local field $\overline{\mathbb{F}_p}((t))$, which is 1-dimensional. Let $\mathcal{O}$ be its ring of integers. Let X be a G-variety, and we may consider X to be a vector space with a group G acting on it. $\mathcal{A}_G(F)$ is a category of $G(F)$-representations. $\Theta_X \in \mathcal{A}_G(F)$ is a function on $X(F)$. For the relative Langlands correspondence, there is a $\check{G}$-variety $\check{X}$ and $\Theta_{\check{X}} \in \mathcal{B}_{\check{G}}(F)$. We want to understand Θ_X as a $G(F)$-representation in a way that is interoperable on the B-side of the Langland correspondence.

For now we aim at understanding the unramified part, where the Hecke algebra $\mathcal{H} :=$ Hecke $(G(F), G(\mathcal{O}))$ acts on the $G(0)$-invariant functions on $X(F)$. We want to diagonalize the action that is compatible with the extra structure from X. Let us first consider a model example.

Example 1: A Model Example

On the A-side, we consider the algebra of compactly supported functions on $\mathbb{Z}$ with the Hecke operator T acting as a shift operator on it. The corresponding B-side is the Laurent polynomial ring $\mathbb{C}[z^{\pm 1}]$. These two algebras are isomorphic.

On the A-side, we pick a function supported only on 0 which is denoted by δ_0. After diagonalization, it gets mapped to the identity $1 \in \mathbb{C}[z^{\pm 1}]$. The action of T on the A-side corresponds to multiplication by z on the B-side. The inner product $\langle\ ,\ \rangle$ on the A-side is defined as

$$\langle f, f\rangle = \int_{S^1} |f|^2 \frac{\mathrm{d}\theta}{2\pi}$$

where f is a function from the A-side. Also, f corresponds to $\sum_n \langle f, z^n\rangle z^n \in \mathbb{C}[z^{\pm 1}]$ on the B-side by a Fourier transform, and reversely $g \in \mathbb{C}[z^{\pm 1}]$ corresponds to $\sum_n \int_{S^1} g(z) z^n$ on the A-side by an inverse Fourier transform.

Take $X = \mathbb{A}^1, G = \mathbb{G}_m$, then we have $\mathbb{A}^1(F) = F$, and a valuation map

$$val : F/\mathcal{O}^* \xrightarrow{\sim} +\infty \sqcup \mathbb{Z}$$

which is an isomorphism. The A-side is $\mathcal{A}_G(F) = C_c(X(F))^{G(\mathcal{O})}$, and the B-side is $\mathbb{C}[\check{G}] = \mathbb{C}[\mathbb{C}^*] = \mathbb{C}[z^{\pm 1}]$. More explicitly,

$$C_c(X(F))^{G(\mathcal{O})} = \{f : bZ \longrightarrow bC : f(n) = \text{ constant }, n \gg 0, f(n) = 0, n \ll 0\}$$

$\mathcal{H} = \mathbb{C}[T^{\pm 1}]$, T is an element of $\mathbb{G}_m(F)$ of evaluated at 1.

$$\delta_0 = 1_{X(\mathcal{O})} = \{f = 1, n \geq 0, f = 0, n < 0\}$$

$$\langle f, g\rangle := \int_F f(x)\overline{g(x)}\mathrm{d}x = \sum f(n)\overline{g(n)}q^{-n}$$

T acts on f by $Tf(n) = f(n+1)q^{-1/2}$. If we use the same Fourier transform $\langle f, z^n \rangle$ as we already mentioned, we will see that $\delta_0 \in C_c(X(F))^{G(\mathcal{O})}$ gets mapped to $\frac{1}{1-q^{-1/2}z} \neq 1 \in \mathbb{C}[z^{\pm 1}]$. To force δ_0 gets mapped to 1, we need to do a rescaling as follows:

We construct

$$e_z : n \mapsto q^{n/2} z^n (1 - q^{-1/2} z) \in C_c(X(F))^{G(\mathcal{O})}$$

Then $\delta_0 \in C_c(X(F))^{G(\mathcal{O})}$ gets mapped to $\langle \delta_0, e_z \rangle = 1 \in \mathbb{C}[z^{\pm 1}]$, and $f \in C_c(X(F))^{G(\mathcal{O})}$ gets mapped to

$$\langle f, e_z \rangle = \int_{S^1} |f(\theta)|^2 \mathrm{d}\mu = \int \frac{1}{(1 - q^{-1/2} e^{i\theta})(1 - q^{-1/2} e^{-i\theta})} \frac{\mathrm{d}\theta}{2\pi} \in \mathbb{C}[z^{\pm 1}]$$

That was the model example. Then we would like to investigate what might happen in a more general case, for example a spherical variety X. On the A-side, we have $C_c(X(F))^{G(\mathcal{O})}, \delta_0, \langle\ ,\ \rangle$, where the Hecke algebra acts on $C_c(X(F))^{G(\mathcal{O})}$; on the B-side, we have $\mathbb{C}[Z]$ where Z lives above $\check{G}/\text{conj}$, that is, there is a map: $Z \to \check{G}/\text{conj}$. The Hecke operator on the A-side can be diagonalized and corresponds to a multiplication on the B-side, which implies that each $z \in Z$ gives a character $\mathcal{H} \to \mathbb{C}$. We also have that $\delta_0 \in C_c(X(F))^{G(\mathcal{O})}$ corresponds to $1 \in \mathbb{C}[Z]$ and $\langle f, f \rangle = \int |f|^2 \mathrm{d}\mu$.

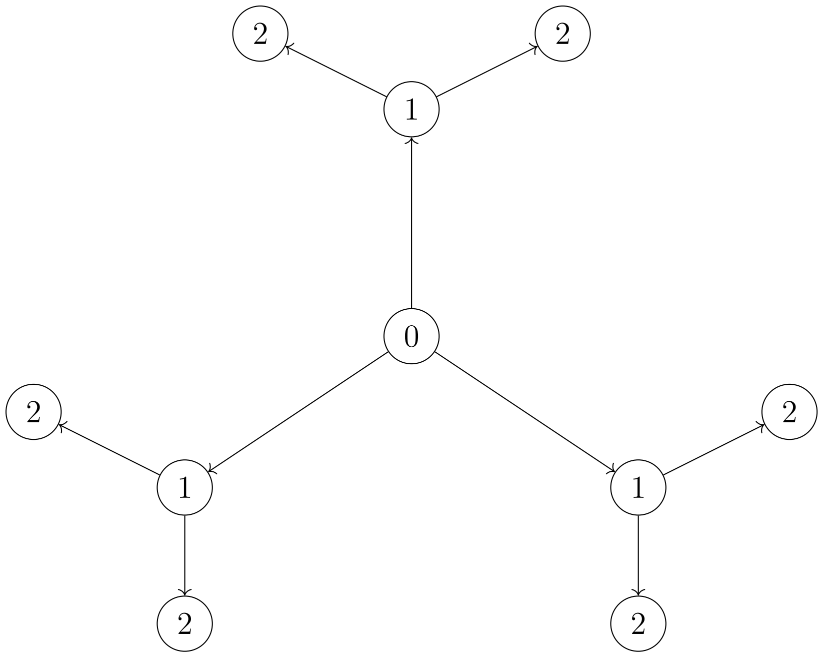
Example 2

Let us consider $G = PGL_2^l \times PGL_2^r$ acting on $X = PGL_2$ where l denotes the left action and r denotes the right action. The Hecke algebra is $\mathcal{H} = \mathbb{C}[T_l, T_r]$. To understand $X(F)/G(\mathcal{O}) = PGL_2(\mathcal{O}) \backslash PGL_2(F)/PGL_2(\mathcal{O})$, we first observe that $PGL_2(F)/PGL_2(\mathcal{O})$ is pictured by vertices of a $q+1$-valent tree as shown in the picture. The Hecke algebra $\mathcal{H}(PGL_2)$ acts on its right, and the Hecke operator T acts on its function f by

$$Tf(x) = \sum_{y \sim x} f(y)$$

where x is a vertex of the tree, and $y \sim x$ denotes its adjacent vertex y. Then as for $PGL_2(\mathcal{O}) \backslash PGL_2(F)/PGL_2(\mathcal{O})$, we have the isomorphism:

$$X(F)/G(\mathcal{O}) = PGL_2(\mathcal{O}) \backslash PGL_2(F)/PGL_2(\mathcal{O}) \simeq \mathbb{Z}_{\geq 0}$$



Therefore functions on $X(F)/G(\mathcal{O})$ are just the radial functions on the tree. It has been shown that $T_l = T_r$, we denote it by T. The action of T is pictured by fixed a center vertex and spin the other vertices around the center. The action of T on the function f is given by

$$\begin{aligned} Tf(0) &= (q+1)f(1), \\ Tf(1) &= f(0) + qf(2), \\ Tf(n) &= f(n-1) + qf(n+1). \end{aligned}$$

We have the following correspondence:

$$C_c(X(F))^{G(\mathcal{O})} \longrightarrow \mathbb{C}[Sl_2/\text{conj}]$$

The LHS is A-side and the RHS is B-side. Since $T_l = T_r = T$ (the Hecke operators coincide), on the B-side Sl_2/conj is in the diagonal of $Sl_2 \times SL_2/\text{conj}$.

We can check that 1 on the A-side gets mapped to 1 on the B-side; the Hecke operators on the A-side gets mapped to the multiplication on the B-side; and

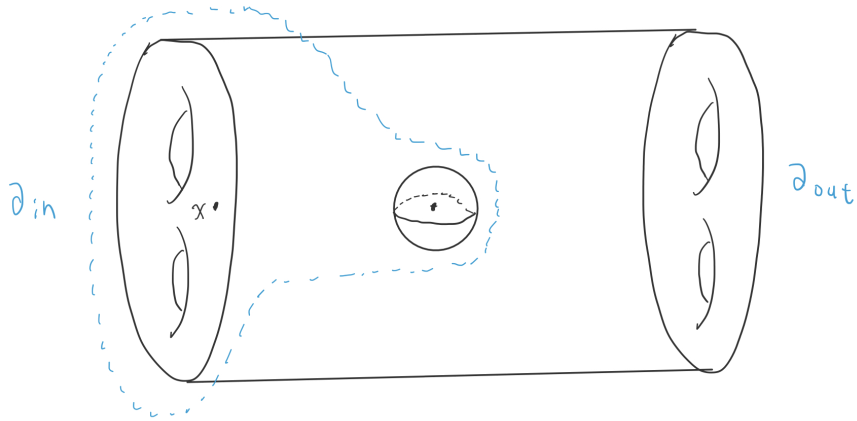
$$\langle f, f\rangle = \int_{SU_2/\text{conj}} f \cdot (q- \text{ character of } SL_2) \cdot \text{ Haar measure}$$

where SU_2 is the maximal compact subgroup of SL_2.

7. Day 2 Lecture 4

We will mainly work with a curve Σ over $\overline{\mathbb{F}}_q$, which is a 2-dimensional arithmetic object from our point of view. For a curve Ξ over $\mathbb{F}_q$ (which is a 3-dimensional arithmetic object), we can get similar statements by using $\Sigma \times S^1$.

In Day 2 Lecture 1, we described how to turn $\mathcal{A}_G(\Sigma)$ into $\mathcal{A}_G(S^2)$-module (or an **actegory**, if one prefers). Let's remind ourselves about the construction. Pick a point $x \in \Sigma$ and delete a small ball with center at $(x, \frac{1}{2})$ from $\Sigma \times [0,1]$. We can now view this as a cobordism from $\Sigma \sqcup S^2$ to Σ.



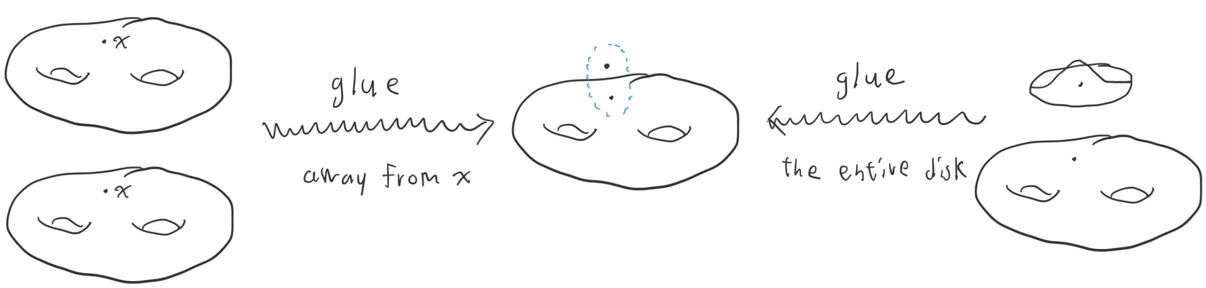
Now we can apply $\mathcal{A}_G$ to get a functor $\mathcal{A}_G(S^2) \otimes \mathcal{A}_G(\Sigma) \to \mathcal{A}_G(\Sigma)$. Since $\mathcal{A}_G(S^2)$ comes with a monidal structure (introduced in Day 1 Lecture 1), the map $\mathcal{A}_G(S^2) \otimes \mathcal{A}_G(\Sigma) \to \mathcal{A}_G(\Sigma)$ equips $\mathcal{A}_G(\Sigma)$ the structure of a $\mathcal{A}_G(S^2)$-module.

When $\mathcal{F}_G(\Sigma) = \mathrm{Bun}_G(\Sigma)$ we have an issue: Bun_G is an algebro-geometric object, but $\Sigma \times [0,1]$ is a topological space with no structure of an algebraic variety. To fix this, let's go further by collapsing the interval in our picture, leaving only a small "bubble" inside Σ (corresponding to the open sphere we removed earlier).

More formally, this is the non-separated algebraic variety "Σ with a double point". It can be equivalently described in either one of the following ways:

$$\Sigma \sqcup_{\Sigma \setminus x} \Sigma = \Sigma \sqcup_D (D \sqcup_{D^\times} D).$$

Here $D = \operatorname{Spec} \overline{\mathbb{F}}_q[[t]]$ is the formal disk, $D^\times = \operatorname{Spec} \overline{\mathbb{F}}_q((t))$ is a punctured formal disk, both centered around x. We will call $D \sqcup_{D^\times} D$ a **raviolo**, denoted by $\mathcal{R}$ for simplicity.



Note that the description $\Sigma \sqcup_D (D \sqcup_{D^\times} D)$ now makes sense in algebraic geometry. We get the **Hecke correspondence**

$$\begin{tikzcd}
& \mathrm{Bun}_G(\Sigma \sqcup_{\Sigma \setminus x} \Sigma = \Sigma \sqcup_D \mathcal{R}) \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G(\Sigma) \times \mathrm{Bun}_G(\mathcal{R}) & & \mathrm{Bun}_G(\Sigma)
\end{tikzcd}$$

given by restriction of G-bundles to the appropriate part of the variety. (The vertex $\mathrm{Bun}_G(\Sigma \sqcup_D \mathcal{R})$ is also called the **Hecke stack**.) This is expected: cobordism between Σ_1 and Σ_2 should give correspondence between $\mathcal{F}_G(\Sigma_1)$ and $\mathcal{F}_G(\Sigma_2)$. Linearizing this, we obtain a map

$$\mathcal{A}_G(\mathcal{R}) \otimes \mathcal{A}_G(\Sigma) \to \mathcal{A}_G(\Sigma),$$

that is, a functor in the 2-dimensional case and an algebra homomorphism if we take 3-dimensional Ξ instead of Σ. Recall that $\mathcal{A}_G(\Sigma) = \mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$, where Shv are some version of constructible sheaves: de Rham, Betti, or étale.

To understand Langlands, we will first have to understand $\mathcal{A}_G(\mathcal{R})$, often called the category of **Hecke operators** (or **Hecke category** for short). For this, we will need the goemetric Satake.

Remark 7.1. In Day 1 Lecture 1, we had a monoidal structure on $\mathcal{A}_G(S^2)$ that turned $\mathcal{A}_G(\Sigma)$ into an $\mathcal{A}_G(S^2)$-module. We can do the same witht he raviolo. Consider now a raviolo with three stacked points in the center

$$X = D_1 \sqcup_{D^\times} D_2 \sqcup_{D^\times} D_3.$$

This is a (nonseparated) variety with three natural maps p_{12}, p_{13}, p_{23} to the raviolo $\mathcal{R}$ (by colliding any two of the three stacked points). We define $m\colon \mathcal{A}_G(\mathcal{R}) \otimes \mathcal{A}_G(\mathcal{R}) \to \mathcal{A}_G(\mathcal{R})$ by $m = p_{13,*}(p_{12}^* \otimes p_{23}^*)$. (As usual, some $*$ here may be !.) Of course, we are omitting some detail here, as the pushforward at the end does not make sense on the nose (it has domain $\mathcal{A}_G(X)$ instead of $\mathcal{A}_G(X) \otimes \mathcal{A}_G(X)$).

Similarly to the case of honest S^2, it can be shown that $\mathcal{A}_G(\mathcal{R})$ is an E_3-category (there are algebro-geometric technical details). In particular, $\mathcal{A}_G(\mathcal{R})$ is a symmetric monoidal category.

It is also possible to add ramifications to the picture. The action is the same: cutting a ball in the middle. The algebra/category at unramified points is the same. However, at a ramified point $x_0 \in \Sigma$ the sphere cannot leave the interval $x_0 \times [0,1]$, so we do not have a E_3-structure.

Let's introduce some more notations: Denote $F = \overline{\mathbb{F}}_q((t))$, and $O = \overline{\mathbb{F}}_q[[t]]$ the ring of integers of F.

Let us describe $\mathcal{A}_G(\mathcal{R}) = \mathrm{Shv}(\mathrm{Bun}_G(\mathcal{R}))$ using double cosets. Recall that $\mathcal{R} = D_1 \sqcup_{D^\times} D_2$. A G-bundle (up to isomorphism) on $\mathcal{R}$ is given by the data of a G-bundle on each D_i, plus a *clutching function* $g : D^\times = D_1 \cap D_2 \to G$. Bundles on D_i are always trivial, and a clutching function is an element in $G(F)$ by definition. Also, two clutching functions g, g' define isomorphic G-bundles if and only if $g' = h_1 g h_2^{-1}$ for some $h_1, h_2 \in G(O)$. Therefore, we have just proved that

Lemma 7.2. $\mathrm{Bun}_G(\mathcal{R}) = G(O)\backslash G(F)/G(O)$.

Remark 7.3. Assume G is semisimple. Using the (harder) fact that any principal G-bundles on a projective Σ trivializes away from a point (due to Harder), one can adapt the argument to the general $\mathrm{Bun}_G(\Sigma)$. This will not be relevant to this lecture.

Note that $G(F)/G(O) = LG/L^+G = \mathrm{Gr}$ is the **affine Grassmannian** of G by definition. Hence $\mathrm{Bun}_G(\mathcal{R}) = G(O)\backslash \mathrm{Gr} = \underline{\mathrm{Gr}}$ is the "$G(O)$-equivariant affine Grassmannian" of G. Therefore, the Hecke category $\mathcal{A}_G(\mathcal{R})$ is

$$\mathcal{H} = \mathrm{Shv}(\underline{\mathrm{Gr}}) = \mathrm{Shv}^{G(O)}(\mathrm{Gr}),$$

the (derived) category of $G(O)$-equivariant constructible sheaves on affine Grassmannian. In the 3-dimensional case, we will get the **Hecke algebra** $H = \mathrm{Fun}(\underline{\mathrm{Gr}}(\mathbb{F}_q))$.

In Day 2 Lecture 2 we discussed the classical Satake isomorphism:

$$\mathrm{Fun}(\underline{\mathrm{Gr}}/\mathbb{F}_q) \cong K_0(\mathrm{Rep}\,\check{G}) \cong \mathcal{O}(\check{G}/\check{G}) = \mathcal{O}(\mathrm{Maps}(S^1, */\check{G})).$$

We now proceed to categorify the isomorphism, leading to first the geometric Satake correspondence, and then the derived geometric Satake.

Consider the abelian category of $G(\mathcal{O})$-equivariant perverse sheaves on the affine Grassmannian, which we denote by $\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit}$. Again, we can work with any sheaf theory: étale, Betti, or de Rham. In the literature, the notation $\mathrm{Perv}(\underline{\mathrm{Gr}})$ is also used. (In fact, by definition, $\mathrm{Perv}(\underline{\mathrm{Gr}})$ is the **heart** of the t-structure on the derived category of $G(O)$-equivariant constructible sheaves $\mathrm{Shv}(\underline{\mathrm{Gr}})$.)

Theorem 7.4 (**Geometric Satake correspondence**)**.** *There is an equivalence of symmetric monoidal categories*

$$\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit} \cong \mathrm{Rep}(\check{G}),$$

where the monoidal structure on $\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit}$ *(resp,* $\mathrm{Rep}(\check{G})$*) is the convolution product of perverse sheaves (resp, tensor products of representations).*

Remark 7.5.

(1) Note that both categories come equipped with natural functors to vector spaces: the forgetful functor

$$\mathrm{Rep}(\check{G}) \to \mathrm{Vect},$$

and the cohomology functor

$$H^*_{G(O)}(\mathrm{Gr}, -) : \mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit} \to \mathrm{Vect}$$

(defined using the t-structure). They are called **fiber functors**, and the Satake equivalence is compatible with these two functors. But the Koszul's rule of sign will be necessary if we use GrVect (graded vector spaces) instead of Vect.

(2) One recovers the classical Satake isomorphism by passing the equivalence to the level of Grothendeick rings K_0. (Or by taking the trace of the identity functors.)

(3) Being an equivalence of symmetric monoidal categories means, in particular, that the monoidal structure on $\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit}$ is commutative. Here is a direct explanation. We saw that $\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit}$ is an E_3-category. An E_n-structure on a category consists of multiplications labeled by the configuration space of points in $\mathbb{R}^n$. As n grows, the multiplications become more and more commutative. For $n = 1$, we get monoidal categories. For $n = 2$, these are braided monoidal categories (i.e. $a \otimes b$ is isomorphic to $b \otimes a$, but these isomorphisms are not necessarily involutive). Finally, when $n = 3$, the first multiplication necessarily becomes commutative, so we get symmetric monoidal categories.

By the geometric Satake, for any point $x \in \Sigma$, we have the (restircted) Hecke action of $\mathrm{Shv}(\underline{\mathrm{Gr}})^{\heartsuit} \cong \mathrm{Rep}\,\check{G}$ on $\mathcal{A}_G(\Sigma) = \mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$. Since the choice of $x \in \Sigma$ is arbitrary, to make things more canonical, we would have to account for all the points. To do this, we can introduce an adelic version of $\mathrm{Rep}(\check{G})$: the restricted tensor product $\bigotimes_{x\in\Sigma}^{\mathrm{restr}} \mathrm{Rep}\,\check{G}$.

The action above is, in general, mysterious, so one hopes to "diagonalize" it. For this, we turn to the action of $\bigotimes_{x\in\Sigma}^{\mathrm{restr}} \mathrm{Rep}\,\check{G}$ on $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$. Once again, any version of local systems (de Rham, Betti, étale) will work just fine.

Convention: In this lecture at least, the coefficient field of our spectral theory (i.e. the naive $\mathcal{B}$-theory, in what we call below) is taken to be $\mathbb{C}$.

To describe the action more precisely, first fix some $x \in \Sigma$. There is an identification

$$\mathrm{Loc}_{\check{G}}\,\Sigma = \mathrm{Map}(\Sigma, */\check{G}) \quad \text{(locally constant maps)}.$$

Then define $\mathrm{ev}_x : \mathrm{Loc}_{\check{G}}\,\Sigma \to */\check{G}$ by evaluating at x. (Or equivalently, we can restrict any $\check{G}$-local system to the point $x \in \Sigma$, which gives a map $\mathrm{Loc}_{\check{G}}\,\Sigma \to */\check{G}$.) On the other hand, every representation defines a vector bundle over a point (or $*/\check{G}$ to be precise), so there is an identification

$$\mathrm{QCoh}(*/\check{G}) = \mathrm{Vect}(*/\check{G}) = \mathrm{Rep}\,\check{G}.$$

Thus, given any representation $V \in \mathrm{Rep}(\check{G})$, we can pullback along ev_x to obtain a quasicoherent sheaf $\mathrm{ev}_x^* V$ on $\mathrm{Loc}_{\check{G}}\,\Sigma$, and the action of V on $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$ is given by the tensor product with $\mathrm{ev}_x^* V$. This extends to a genuine action of $\bigotimes_{x\in\Sigma}^{\mathrm{restr}} \mathrm{Rep}\,\check{G}$.

Remark 7.6. If Σ is defined over $\mathbb{F}_q$, to any $x \in \Sigma(\mathbb{F}_q)$ we can furthermore associate a map $\mathrm{Loc}_{\check{G}}\,\Sigma \to \check{G}/\check{G}$. This is because the Frobenius morphism plays the role of S^1.

Naively, one can propose the geometric Langland's correspondence as:

$$\text{Are } \mathrm{Shv}(\mathrm{Bun}_G(\Sigma)) \text{ and } \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma) \text{ equivalent (as } \left(\bigotimes_{x\in\Sigma}^{\mathrm{restr}} \mathrm{Rep}\,\check{G}\right)\text{-modules)?}$$

This turns out to be negative. The key reason is that while the action of $\bigotimes_{x\in\Sigma}^{\mathrm{restr}} \mathrm{Rep}\,\check{G}$ on $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$ is locally constant, the action on $\mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$ is not at all locally constant.

In order for there to be an equivalence, we need to either replace the right-hand side with $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$ (**Ind-coherent sheaves**), or the left-hand side with **tempered sheaves**.

Remark 7.7. For each version of sheaf theory (de Rham, Betti, and étale), there is a spectral action theorem, and they all can be roughly summarized as the following.

Theorem 7.8 (**Spectral action**). *The action of* $\bigotimes_{x\in\Sigma}^{restr} \mathrm{Rep}\,\check{G}$ *on* $\mathcal{A}_G(\Sigma) = \mathrm{Shv}(\mathrm{Bun}_G(\Sigma))$ *factors through the* $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$*-action.*

Nonetheless, let us (for now) naively study $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$. Thus we consider the $\mathcal{B}$-theory, which to a 2-manifold Σ assigns $\mathcal{B}_{\check{G}}(\Sigma) = \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}\,\Sigma)$, and to a 3-manifold Ξ, the vector space $\mathcal{B}_{\check{G}}(\Xi) = \mathcal{O}(\mathrm{Loc}_{\check{G}}\,\Xi)$. Unlike the $\mathcal{A}$-theory, this defines an actual TQFT. That is, the output only depends on the topology of Σ or Ξ.

Just like the $\mathcal{A}$-theory, we should first take some time to look at $\mathcal{B}_{\check{G}}(\mathcal{R})$. Since the raviolo is just a disk with a double point, we can identify local systems on $\mathcal{R}$ as a pair of local systems on the disk D^2 together with an identification of them on $S^1 = \partial D^2$. Thus, we can compute

$$\mathrm{Loc}_{\check{G}}(\mathcal{R}) = \mathrm{Loc}_{\check{G}}(D^2) \times_{\mathrm{Loc}_{\check{G}}(S^1)} \mathrm{Loc}_{\check{G}}(D^2),$$

If we temporarily choose Betti as our sheaf theory, then we get the identifications (of stacks)

$$\mathrm{Loc}_{\check{G}}(D^2) = \{\pi_1(D^2) \to \check{G}\}/\check{G} = */\check{G},$$

$$\mathrm{Loc}_{\check{G}}(S^1) = \{\pi_1(S^1) \to \check{G}\}/\check{G} = \check{G}/\check{G}$$

(where in both cases $\check{G}$ acts by conjugation). They lead to the identification

$$\mathrm{Loc}_{\check{G}}(\mathcal{R}) = */\check{G} \times_{\check{G}/\check{G}} */\check{G} = (* \times_{\check{G}} *)/\check{G}$$

Here the point $*$ is included into $\check{G}$ as the identity element. Hence we are using only a neighborhood of identity when taking the fiber product $* \times_{\check{G}} *$. It follows that we can replace $\check{G}$ with its Lie algebra $\check{\mathfrak{g}}$ and obtain the identification:

$$* \times_{\check{G}} * = * \times_{\check{\mathfrak{g}}} * = \Omega\check{\mathfrak{g}}$$

where $\Omega\check{\mathfrak{g}}$ is the loop space of $\check{\mathfrak{g}}$.

For a general vector space V, $* \times_V *$ is the self-intersection of the point $*$. For the usual algebraic varieties, we would compute

$$* \times_V * = \mathrm{Spec}\,\mathbb{C} \times_{\mathrm{Spec\ Sym}\,V^*} \mathrm{Spec}\,\mathbb{C} = \mathrm{Spec}\,(\mathbb{C} \otimes_{\mathrm{Sym}\,V^*} \mathbb{C}).$$

But we are working in a derived context, which means we should take derived tensor product in the last step. Writing Koszul complex we see that $\mathbb{C} \otimes^L_{\mathrm{Sym}\,V^*} \mathbb{C}$ is an exterior algebra in $\dim V$ variables of degree 1. It can be checked that the multiplication on both sides is the same and we have a natural isomorphism

$$\mathbb{C} \otimes^L_{\mathrm{Sym}\,V^*} \mathbb{C} = \mathrm{Sym}\,V^*[1].$$

Here 1 is a cohomological shift, in particular $\operatorname{Sym} V^*[1]$ is an exterior algebra.

Hence approximately, $\mathcal{B}_{\check{G}}(\mathcal{R})$ is $\operatorname{QCoh}(\operatorname{Loc}_{\check{G}}(\mathcal{R})) = \operatorname{Sym} \check{\mathfrak{g}}^*[1]\text{-Mod}^G$. Note that the monoidal structure is not the tensor product of modules; it is convolution coming from the fact that

$$\operatorname{Loc}_{\check{G}}(\mathcal{R}) = */\check{G} \times_{\check{G}/\check{G}} */\check{G}$$

is a fibered product of two copies of the same stack. This can be checked using the pair of pants definition of the monoidal structure.

Remark 7.9 (**Koszul duality**)**.** Given an exterior k-algebra Λ equipped with an **augmentation** $\epsilon : \Lambda \to k$, consider the functor $\operatorname{Hom}_\Lambda(k, -) : \Lambda\text{-Mod} \to \operatorname{End}_\Lambda(k)\text{-Mod}$. Here

(1) the monoidal structure on Λ-Mod is given by convolution product: given any two Λ-modules, push them forward to k-modules via ϵ, take tensor product, and then pull it back via ϵ;
(2) the $\operatorname{End}_\Lambda(k)$-action on $\operatorname{Hom}_\Lambda(k, M)$ is given by precomposition with endomorphisms.

This functor is not an equivalence of derived categories of modules unless we carefully specify what kinds of modules we're considering. In fact, it restricts to an equivalence between *coherent* Λ-modules and *perfect* $\operatorname{End}_\Lambda(k)$-modules.

The Koszul duality above can be upgraded to the category of local operators, establishing an equivalence

$$\operatorname{Coh}(\operatorname{Loc}_{\check{G}}(\mathcal{R})) \cong \operatorname{Perf}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

One can ind-complete these categories and get an equivalence between

$$\operatorname{IndCoh}(\operatorname{Loc}_{\check{G}}(\mathcal{R})) \cong \operatorname{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

This is an equivalence of *monoidal* categories: The RHS has the usual monoidal structure, and the corresponding monoidal structure on the left is the convolution operation coming from the groupoid structure on $\Omega\check{\mathfrak{g}}$.

Coming back to our $\mathcal{A}$- and $\mathcal{B}$-theories, we are ready to state the derived version of the geometric Satake equivalence (one without the heart). We have naively asked for an equivalence between $\mathcal{A}_G(\mathcal{R}) = \operatorname{Shv}(\underline{\operatorname{Gr}})$ and $\mathcal{B}_{\check{G}}(\mathcal{R}) = \operatorname{QCoh}(\operatorname{Loc}_{\check{G}} \mathcal{R})$. As we have said, this is not correct, and the correct version is the following:

Theorem 7.10 (**Derived Geometric Satake**)**.** *There is an equivalence of symmetric monoidal categories*

$$\operatorname{Shv}(\underline{\operatorname{Gr}}) \cong \operatorname{IndCoh}(\Omega\check{\mathfrak{g}}/\check{G}) = \operatorname{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G}).$$

Remark 7.11.

(1) As a heads up, on the RHS of the derived Satake, the coadjoint representation shows up, which is one of the few indications that the moment map (and hence a Hamiltonian action) will be relevant in our story. This will become a big theme in subsequent lectures.

(2) Similar to the geometric Satake, the categories in the derived Satake also come with fiber functors. On the affine Grassmannian side, we can take equivariant cohomology

$$H^*_{G(O)}(\underline{\mathrm{Gr}}, -) : \mathrm{Shv}(\underline{\mathrm{Gr}}) \to H^*_{G(O)}(\mathrm{pt})\text{-Mod}$$

On the other side, there is a functor

$$\mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/\check{G}) \to \mathcal{O}(\check{\mathfrak{g}}^*[2]/\check{G})\text{-Mod}$$

given by restriction of **Kostant slices** (which we will not explain here). It follows from some computations that

$$\begin{aligned}\mathcal{O}(\check{\mathfrak{g}}^*[2]/\check{G}) = \mathcal{O}(\check{\mathfrak{g}}^*[2])^{\check{G}} = \mathrm{Sym}\,\check{\mathfrak{t}}[2]^W &= \mathrm{Sym}\,\mathfrak{t}^*[2]^W \\ &= \mathrm{Sym}\,\mathfrak{t}[-2]^W = \mathrm{Sym}\,\mathfrak{g}[-2]^G = H^*_{G(O)}(\mathrm{pt})\end{aligned}$$

And again, the equivalence is compatible with these two fiber functors.

8. Day 3 Lecture 1

We have not yet spoken about spherical varieties, which have an important role in the BZSV story. Rather than defining them at the outset, we will motivate them by thinking about the kinds of varieties that permit us to generalize the harmonic analysis that we have already seen.

Let G be a split reductive group over a non-archimedean field F. Let $\mathcal{O}$ be the valuation ring with residue field $\mathbf{F}_q$. Let

$$\mathcal{H}_G := \mathcal{H}(G(F), G(O))$$

denote the Hecke algebra, viewed as the space of compactly supported $G(\mathcal{O})$ bi-invariant functions on $G(F)$, with multiplication given by convolution with respect to the Haar measure on G. By the Satake isomorphism, we have an isomorphism

$$\mathcal{H}_G \xrightarrow{\sim} \mathbf{C}[\mathrm{Rep}\,\check{G}], \quad f \mapsto \hat{f}$$

where the right hand side means the group algebra over the Grothendieck ring K_0 of the tensor category $\mathrm{Rep}_{\mathbf{C}}\,\check{G}$.

Now, we upgrade. Let X denote a smooth affine space equipped with a G action. We may consider a space of compactly supported, locally constant functions $C_c^\infty(X(F))$, wich we call the "Schwartz space" for X, and denote $\mathcal{S}(X(F))$. This is naturally a $G(F)$ representation. We consider the space of right-$G(\mathcal{O})$ invariants

$$\mathcal{S}(X(F))^{G(\mathcal{O})} := C_c^\infty(X(F))^{G(\mathcal{O})},$$

which is a module for the right action of the Hecke algebra $\mathcal{H}_G$. (This is called the "spherical" Schwartz space of $X(F)$.) But we have also seen that we really should consider this vector space as an inner-product space:

$$\mathcal{S}(X(F))^{G(\mathcal{O})} \subset (L^2(X(F)), \langle , \rangle)$$

where we integrate products with respect to an invariant measure on $X(F)$. (Alternatively, we may consider our L^2 space to consist of half-densities on $X(F)$, in which case we would write a functions f in our spherical Hecke algebra as $f|dx|^{1/2}$.) We denote

$$\mathbf{1}_X := \mathbf{1}_{X(\mathcal{O})}$$

the characteristic function of $X(\mathcal{O})$. This is our "basic vector" in the Schwartz space. (In general, this formula for the basic function works well when X is smooth. For non-smooth varieties, which are outside of our scope here, the basic vector should be related to the IC sheaf of the loop space of X.)

In previous lectures and exercises, we already encountered two examples of inner products in the spherical Schwartz space:

Example 8.1. Let $X = \mathbf{A}^1$ and $G = \mathbf{G}_m$ act by scaling. The set of $\mathbf{G}(\mathcal{O})$-orbits in $\mathbf{A}^1$ are simply the annuli $\{x \in \mathbf{A}^1(F) : |x| = q^{-n}\}_{n\in\mathbf{Z}}$, and the spherical Schwartz space consists of finite combinations of indicator functions of these annuli. The Hecke algebra is $\mathcal{H}_G \simeq \mathbf{C}[\mathbf{Z}] \xrightarrow{\sim} \mathbf{C}[\mathbf{C}^*]$. (An unfortunate conflict of notation is inevitable here: $\mathbf{C}[\mathbf{Z}]$ denotes the *group algebra* of $\mathbf{Z}$, while $\mathbf{C}[\mathbf{C}^*]$ denotes algebraic functions on $\mathbf{C}^* = \mathbf{G}_m$ over $\mathbf{C}$.) We see that the Satake transform is a kind of Fourier transform, which we will denote $h \in \mathcal{H}_G \mapsto \hat{h} \in \mathbf{C}[\mathbf{C}^*]$. Then the inner product in $\mathcal{S}(X(F))$ is given as follows in terms of the Satake transform:

$$\langle h \star \mathbf{1}_X, \mathbf{1}_X\rangle = \int_{S^1\subset\mathbf{C}^*} \frac{\hat{h}(z)}{(1-q^{-1/2}z)(1-q^{-1/2}z^{-1})} d^\times z \tag{8.2}$$

where $d^\times z = \frac{dz}{z}$ is the multiplicative Haar measure. Notice that we have seen the q-character in the homework.

Definition 8.3. The quantity $\frac{d^\times z}{(1-q^{-1/2}z)(1-q^{-1/2}z^{-1})}$ is called the *Plancherel density of* $\mathbf{1}_{\mathbf{A}^1}$.

Let us recall how this q-character arises on the "dual" (aka the $\mathcal{B}$)-side. In general, $M = T^*X$, so in this case we have $M = T^*\mathbf{A}^1$. Then, dually, we have $\check{G} = \mathbf{G}_m$, which acts on $\check{M}$, which also happens to equal $T^*\mathbf{A}^1$. The action occurs with weight $(1, -1)$; namely, $\lambda.(x, y) \mapsto (\lambda x, \lambda^{-1}y) \in T^*\mathbf{A}^1 = \mathbf{A}^1 \times (\mathbf{A}^1)^*$. (Here we have an exceptional auto-duality on $\mathbf{G}_m$ and $T^*\mathbf{A}^1$; in general $\check{G}$ and $\check{M}$ will be quite different from G and M.)

Recall that the q-character is defined by

$$q\text{-tr}(z) = \sum_{i\geq 0} q^{-i/2}\mathrm{tr}(z|\mathbf{C}[\check{M}]_i) = \frac{1}{\det(I - q^{-1/2}z|_{\check{M}})}, \tag{8.4}$$

where $\mathbf{C}[\check{M}]_i = \mathrm{Sym}^i(\check{M}^\vee) = \mathrm{Sym}^i(\check{M})$ (by the linear self-duality of $\check{M} = \mathbf{A}^1 \times (\mathbf{A}^1)^*$ as a vector space). The grading here comes from what we shall call the $\mathbf{G}_{gr}$ action, which scales both $\mathbf{A}^1$ factors of $\check{M}$ equally (i.e., $\mathbf{G}_{gr}$ acts with weights $(1, 1)$). Because $\mathbf{G}_{gr}$ is isomorphic to $\mathbf{G}_m$, this may lead to some confusion; so we emphasize here that there are two different actions of $\mathbf{G}_m$ on $\check{M}$: the action of $\check{G} = \mathbf{G}_m$ and the action of $\mathbf{G}_{gr} = \mathbf{G}_m$.

In summary, we see that the expression $\frac{1}{(1-q^{-1/2}z)(1-q^{-1/2}z^{-1})}$ is the q-character of $\check{M} = T^*\mathbf{A}^1$ under the $\mathbf{G}_{gr}$-grading. We denote this expression by $L(z, \check{M})$. In general, the Plancherel density has the form

$$\text{Plancherel density} = (q\text{−character of symplectic variety } \check{M})d^\times z = L(z, \check{M})d^\times z,$$

and we may re-write (8.2) as

$$\langle h \star \mathbf{1}_X, \mathbf{1}_X \rangle = \int_{S^1 \subset \mathbf{C}^*} \hat{h}(z) L(z, \check{M}) d^\times z. \tag{8.5}$$

We hope for this formula to hold on a more general class of G-varieties X, with corresponding $M = T^*X$ and duals $\check{G}_X$, $\check{M}$. (Observe that *only* $\check{M}$ appears in the formula for the Plancherel density – in particular, if more than one variety G-variety X gives rise to the same $\check{G}_X$ and M, then the Plancherel density of their basic vectors are expected to agree.)

Remark. (For the arithmetically-minded.) There are a lot of $q^{1/2}$'s in our formulae; this suggests that one must pick a preferred square root of q when working over fields other than $\mathbf{C}$. This particular ambiguity in the arithmetic setting is resolved by using the so-called C-group, rather than the L-group. Over $\mathbf{C}$, of course, we may choose the positive $q^{1/2}$.

Example 8.6. Let $X = H$ and $G = H \times H$, where H is a split reductive group. The bimodule structure is $(h_1, h_2) \in G : x \in X \mapsto h_1^{-1} x h_2$. The Schwartz space is just the Hecke algebra of H:

$$\mathcal{S}(X)^{G(\mathcal{O})} \cong \mathcal{H}_H.$$

On the dual side, we find $\check{G} = \check{H} \times \check{H}$ and $\check{G}_X = \check{H}$. Then we have an anti-diagonal embedding

$$\check{G}_X = \check{H} \xhookrightarrow{(C,\mathrm{Id})} \check{H} \times \check{H} = \check{G}, \quad a \mapsto (a^{-1}, a)$$

where C denotes Cartan involution. Thus, the identification of the Hecke algebra under classical Satake transform becomes:

$$\mathcal{H}_G = \mathcal{H}_H \otimes \mathcal{H}_H \cong \mathbf{C}[\check{H}]^{\check{H}}, \;\; h_1 \otimes h_2 \mapsto \hat{h}_1(z^{-1}) \cdot \hat{h}_2(z).$$

Thus the inner product in this case reads

$$\langle (h_1 \otimes h_2) \star \mathbf{1}_H, \mathbf{1}_H \rangle = \int_{\check{H}_c} \frac{\hat{h}_1(z^{-1})\hat{h}_2(z)}{\det(I - q^{-1}z|_{\check{\mathfrak{h}}})} d_{\mathrm{Haar}}(z) \tag{8.7}$$

$$= \int_{\check{H}_c} \hat{h}_1(z^{-1})\hat{h}(z) L(z, \check{\mathfrak{h}}) d_{\mathrm{Haar}} z. \tag{8.8}$$

Let us clarify the various terms appearing in the integral. We are integrating over a compact form $\check{H}_c$ of the dual group $\check{H}$, and $d_{\mathrm{Haar}(z)}$ denotes the Haar measure. We have $\check{\mathfrak{h}} = \mathrm{Lie}(\check{H})$ and $L(z, \check{\mathfrak{h}}) := \frac{1}{\det(\mathrm{Id} - q^{-1}z)}$, which denotes the q-character of the adjoint action of $\check{H}$ on $\check{\mathfrak{h}}$. Lastly, the grading $\mathbf{G}_{gr}$-action on $\check{\mathfrak{h}}$ is $\lambda.z \mapsto \lambda^2 z$, the squaring of the usual scaling action (hence the appearance of q^{-1} instead of $q^{-1/2}$ in the denominator). Notice that the integrand is *not* $\check{G}_c$; it is $\check{H}_c$, reflecting the fact that in this example, $\check{G}_X = \check{H}$, not $\check{G}$.

As we have remarked, we have a general expectation that for a sufficiently nice class of (G, X) (where $G \circlearrowleft X$), a similar story holds true. Given X, we expect there to exist a dual group $\check{G}_X \subset \check{G}$, with the following property. We have an action $\mathcal{S}(X(F))^{G(\mathcal{O})} \circlearrowright \mathcal{H}_G$, giving us a map

$$\mathcal{H}_G \to \operatorname{End}\left(\mathcal{S}(X(F))^{G(\mathcal{O})}\right) \tag{8.9}$$

Now, under the Satake isomorphism $\mathcal{H}_G \simeq \mathbf{C}[\operatorname{Rep}\check{G}]$, where $\mathbf{C}[\operatorname{Rep}\check{G}_X]$ denotes the group algebra of the Grothendieck ring $K_0(\operatorname{Rep}\check{G}_X)$. A representation of $\check{G}$ naturally restricts to a representation of any subgroup $\check{G}' \subset \check{G}$; in particular, this gives us a natural ring homomorphism $\mathbf{C}[\operatorname{Rep}\check{G}] \xrightarrow{\text{restriction}} \mathbf{C}[\operatorname{Rep}\check{G}']$. We expect that $\check{G}_X$ is a reductive subgroup of $\check{G}$ such that the action of $\mathcal{H}_G$ factors as:

$$\begin{tikzcd}
\mathcal{H}_G \arrow[rr] \arrow[d, "\text{Satake}\,\simeq"'] & & \operatorname{End}_{\mathbf{C}}\left(\mathcal{S}(X(F))^{G(\mathcal{O})}\right) \\
\mathbf{C}[\operatorname{Rep}\check{G}] \arrow[rr, "\text{restriction}"'] & & \mathbf{C}[\operatorname{Rep}\check{G}_X]. \arrow[u]
\end{tikzcd} \tag{8.10}$$

In other words, we expect the action of the Hecke algebra $\mathcal{H}_G \simeq \mathbf{C}[\operatorname{Rep}\check{G}]$ on the spherical Schwartz space to "factor through" its restriction to $\mathbf{C}[\operatorname{Rep}\check{G}_X]$. (Note that this restriction map may not be surjective, however.)

We expect the "basic vector" $\mathbf{1}_{X(\mathcal{O})}$ to generate the whole spherical Schwartz space $\mathcal{S}(X(F))^{G(\mathcal{O})}$ as an $\mathcal{H}_G$-module. In fact, we expect more. Let us write $\mathbf{C}[\operatorname{Rep}\check{G}_X] := \mathcal{H}_G^X$. We expect that the module $\mathcal{S}(X(F))^{G(\mathcal{O})}$ is free of rank one over $\mathcal{H}_G^X$, with a distinguished generator $\mathbf{1}_{X(\mathcal{O})}$. In other words, we may think of the right vertical map of (8.10) as giving a canonical isomorphism:

$$\mathcal{H}_G^X \xrightarrow{\sim} \operatorname{End}_{\mathcal{H}_G}\left(\mathcal{S}(X(F))^{G(\mathcal{O})}\right), \tag{8.11}$$

or, since we have a distinguished vector in $\mathcal{S}(X(F))^{G(\mathcal{O})}$, we have a natural isomorphism of vector spaces

$$\mathcal{H}_G^X \xrightarrow{\sim} \mathcal{S}(X(F))^{G(\mathcal{O})}, \tag{8.12}$$

sending

$$\mathbf{C}[\operatorname{Rep}\check{G}_X] \ni \operatorname{Triv} \mapsto \mathbf{1}_{X(\mathcal{O})} \in \mathcal{S}(X(F))^{G(\mathcal{O})}. \tag{8.13}$$

The multiplication on $\mathcal{S}(X(F))^{G(\mathcal{O})}$ inherited from the LHS under this identification comes from *fusion*. (Indeed, convolution does not make sense on X for X not a group.)

Finally, we note that developing a Tannakian formalism for the algebra $\mathrm{End}_{\mathcal{H}_G}\left(\mathcal{S}(X(F))^{G(\mathcal{O})}\right)$ (or the category of right $G(\mathcal{O})$-equivariant perverse sheaves on $X(F)$ under the above identifications) will lead us to the Gaitsgory-Nadler construction of $\check{G}_X$.

For $h \in \mathcal{H}_G$, we let $\hat{h}$ denote its classical Satake transform, which we view (via its trace) as an algebraic (class) function on $\check{G}$. Our expectation is that:

$$\langle \mathbf{1}_{X(\mathcal{O})} \star h, \mathbf{1}_{X(\mathcal{O})} \rangle = \int_{\check{G}_{X,c} \subset \check{G}_X} \hat{h}(z) L(z, V_X) d_{\mathrm{Haar}} z, \tag{8.14}$$

or more generally,

$$\langle \mathbf{1}_{X(\mathcal{O})} \star h_1, \mathbf{1}_{X(\mathcal{O})} \star h_2 \rangle = \int_{\check{G}_{X,c} \subset \check{G}_X} \hat{h_1}(z) \overline{\hat{h_2}}(z) L(z, V_X) d_{\mathrm{Haar}} z, \tag{8.15}$$

where $\check{G}_{X,c}$ denotes the real points of a compact form of $\check{G}_X \subset \check{G}$, V_X denotes a $\mathbf{G}_{\mathrm{gr}}$-graded representation of $\check{G}_X$, and $L(z, V_X)$ is an appropriate local L-factor generalizing the q-character. Observe that the RHS of (8.15) only sees the values of $\hat{h}$ restricted to $\check{G}_X$ (in fact, to a compact form of $\check{G}_X$) whereas by definition $\hat{h}$ is a function on $\check{G}$.

The need to include the L-factor – which in turn depends upon the $\mathbf{G}_{gr}$-grading – to preserve inner products under the Satake transform, reflects the geometric nature of *derived* Satake, which will be discussed in other lectures. The key idea is that inner products become Hom (or Ext) groups under categorification, and the $\mathbf{G}_{gr}$-grading corresponds to a *shearing* in the derived category of equivariant sheaves on $X(F)$.

But let us turn to the quesiton of which G-varieties might satisfy the above desiderata. It turns out that our expectations *very* much restrict the class of X. Our expectations imply, in particular, that for all characters $\chi : \mathcal{H}_G \to \mathbf{C}$ (or, equivalently, all $\chi \in \check{A}/\!\!/W_G(\mathbb{C})$, where $\check{A}$ is the dual torus),

$$\dim \mathrm{Hom}_{\mathcal{H}_G}(\mathrm{Fun}_c(X(F), \mathbb{C}), \mathbb{C}_\chi) < \infty. \tag{8.16}$$

For example, if we let $X = G$, and consider only the left action of G on itself, we have by Peter-Weyl that

$$\mathrm{Fun}(G(F)) = \int^{\oplus} \pi \otimes \widetilde{\pi}\, d\pi, \tag{8.17}$$

which does *not* satisfy (8.16) for each character (since $\dim(\pi) = \infty$ for most irreps π).

Claim. Assume G is split. Then finiteness condition (8.16) is satisfied if and only if X is a spherical variety.

This is how/why the class of spherical varieties enters the BZSV framework.

Definition 8.18. A normal, irreducible G-variety X is spherical if one (or equivalently any) Borel $B \subset G$ acts with an open dense orbit.

Without loss of generality, we will assume from now on that all spherical varieties are quasi-affine. This does not reduce generality because we may always reduce to this case – at the cost of introducing a commuting action $\mathbb{G}_m^N$ on X. Indeed, say we have a projective embedding $X \to \mathbb{P}^N$. Then we consider the usual surjection $V^* := V \setminus \{0\} \to \mathbb{P}^N$, for $V = \mathbb{A}^{n+1}$, and replace X with $\widetilde{X}$ in the Cartesian diagram:

$$\begin{tikzcd} G \times \mathbb{G}_m^N \circlearrowleft & \widetilde{X} \arrow[r] \arrow[d] & V^* \arrow[d] \\ G \circlearrowleft & X \arrow[r] & \mathbb{P}^N. \end{tikzcd}$$

For example, we may replace the flag variety G/B (with the left G-action) with the basic affine space G/N thought of as a $G \times A$-variety (A the maximal torus).

Theorem 8.19. *X is a (quasi-affine) spherical G-variety iff $F[X]$ (the ring of regular functions of X) is multiplicity-free as a G-module.*

Proof. ($\implies$.) Fix $B \subset G$. We will from now on assume that G acts on the right. We observe that $F[X]$ is a(n infinite-dimensional) representation of G which is locally finite; it thus decomposes as a direct sum of highest-weight modules V_χ, where χ is a character of B (i.e., a character of A pulled back to B along the canonical surjection $B \twoheadrightarrow A$). We wish to show that V_χ has multiplicity 1. Let f_χ denote the highest weight vector in V_χ, thought of as a function on X. Now, because B acts via χ on f_χ, we find that $f_\chi(xb) = \chi(b) f_\chi(x)$ for all $x \in X$, $b \in B$. Thus the value of f_χ on a B-orbit in X is determined by its value at one point. However, X has a *dense* B-orbit by assumption, so f_χ is determined uniquely up to scaling on this dense orbit, and thus on all of X. In other words, any other (B, χ)-eigenvector in $F[X]$, can only differ from f_χ by a scalar, proving that the multiplicity of (B, χ) eigenvectors in $f[X]$ is 1, proving that the multiplicity of V_χ in $f[X]$ is 1, as desired. $\square$

To study a spherical variety we examine 1) the dense B-orbit, denoted $\mathring{X}$, and 2) the complement $X \setminus \mathring{X}$. Firstly, if $x \in \mathring{X}$ is arbitrary, $\mathring{X} \simeq B_x \backslash B$. Note that a different choice of $x \in X$ will lead to a conjugate subgroup of B.

Now, consider the space of rational functions $F(X)$ on X. By density, this is the same as the space of rational functions on $\mathring{X}$. We now consider the collection of B-eigenfunctions in $F(X)$, which we denote by $F(X)^{(B)}$. Observe that such functions are invariant under the (right) action of N. We have:

$$F(X)^{(B)} \simeq F(B_x \backslash B/N)^{(A)}. \tag{8.20}$$

From this we may define A_X, "the Cartan of X," which is given by the largest quotient of A which acts faithfully on $\mathring{X}/N$. Observe that this is independent of choice of B and x, because B and B_x are each determined up to conjugacy and are thus canonically equal mod N in "the" Cartan (i.e., the maximal torus viewed as a canonical subquotient) of G.

Thus we have a surjection

$$A \twoheadrightarrow A_X \tag{8.21}$$

yielding a map

$$\check{A}_X \to \check{A} \tag{8.22}$$

of dual tori. Now, (8.22) is *not, in general an injection!* It is an injection if and only $\ker(A \twoheadrightarrow A_X)$ is connected.

In the next lecture, we will extend $\check{A}_X$ to a reductive group $\check{G}_X$; a subgroup of $\check{G}$ with maximal torus $\check{A}_X$. This will be the desired dual group of the spherical variety discussed above.

Let us briefly talk about 2), the complement of $\mathring{X}$ in X. It is a union of B-invariant divisors. We find that the regular functions on X which are B-eigenfunctions are precisely those elements of $F(X)^{(B)}$ with positive valuation on the B-divisors in this complement. The data of the B-divisors and their pairing (via valuation) with $F(X)^{(B)}$ is key to offering a combinatorial description of spherical varieties.

Example 8.23. Let $X = \mathrm{GL}_n$, acted on by $G = \mathrm{GL}_{n-1} \times \mathrm{GL}_n$ on the left and right respectively. Determining $F[X]$ as a G-module is closely related to the so-called "branching problem" of computing the restriction $\pi_{\mathrm{GL}_n}|_{\mathrm{GL}_{n-1}}$.

9. Day 3 Lecture 2 (by Akshay Venkatesh)

Typed by: Chun-Hsien Hsu, Weixiao Lu

We have mentioned previously that there should be a matching as follows:

$$\begin{array}{ccc} (M = TX) & & (\check{M} = T^*\check{X}) \\ G \curvearrowright X & & \check{G} \curvearrowright \check{X} \\ \Theta_G(X) \in \mathcal{A}_G(Y) & \longleftrightarrow & \Theta_{\check{G}}(\check{X}) \in \mathcal{B}_{\check{G}}(Y) \end{array}$$

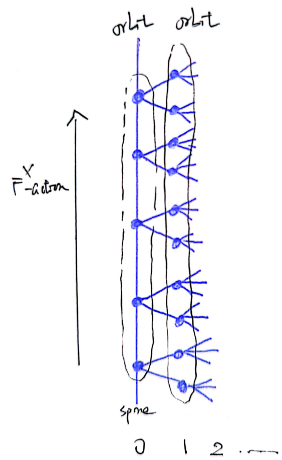
for all arithmetic manifolds Y. The goal of this and the following lecture is to explain the matching for $Y = \overline{\mathbb{F}}_q((t))$ by geometrizing the Plancherel measure/Plancherel formula. We will rewrite the Plancherel measure, so pairings on both sides look like certain hom spaces(twisted by q-characters), which will suggest an equivalence of categories and motivate the matching statement/conjecture. Once we arrive at the matching statement, we will come back and can look in more depth the Plancherel measure.

Let $F = \mathbb{F}_q((t))$ and $\mathcal{O}$ be its ring of integers.

9.1. **An example:** $\mathbb{G}_m \backslash \mathrm{PGL}_2$**.** We start with discussing the Plancherel measure for $X = \mathbb{G}_m \backslash \mathrm{PGL}_2$ (acted on the right by $G = \mathrm{PGL}_2$) as considered in the second exercise from yesterday. First, we need to understand how one should think of

$$X(F)/G(\mathcal{O}) = F^\times \backslash \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}).$$

In the previous lecture, we saw that $\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O})$ can be drawn as a $(q+1)$-valent tree. Now we will draw it in a different way so that one can easily describe the $F^\times$-action on the tree. Let us take $q = 3$, so we are drawing a 4-valent tree:



Then $F^\times$ acts by vertical translation, and orbits in $X(F)/G(\mathcal{O})$ are represented by distances toward the spine, so

$$X(F)/G(\mathcal{O}) \cong \mathbb{Z}_{\geq 0}.$$

Therefore, one can write down Hecke operators as in the case for $\mathrm{PGL}_2(\mathcal{O})\backslash\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O})$ as certain linear recurrence operators. Eventually, in the exercise you will show

$$\begin{aligned} C_c^\infty(X(F))^{G(\mathcal{O})} &\xrightarrow{\sim} \mathbb{C}[\mathrm{SL}_2/\mathrm{conj}] \\ \text{the basic function } \delta_0 = \mathbf{1}_{X(\mathcal{O})} &\mapsto 1 \\ \text{Hecke operators} &\mapsto \text{Multiplication operators} \\ \langle \cdot, \cdot \rangle &\leftrightarrow \int_{\mathrm{SU}_2(\mathbb{R})/\mathrm{conj}} |\cdot|^2 \cdot (q \text{ character of } \check{M} = T^*\mathbb{A}^2)\, d_{\mathrm{Haar}}. \end{aligned} \tag{9.1}$$

Note that SL_2 is the dual group of X (which equals to the dual group of PGL_2 in this case) and $\mathrm{SU}_2 = (\mathrm{SL}_2)_c$ is the compact form of SL_2. What we will do in a moment is to rewrite the inner product and the integral geometrically as a measurement of hom spaces between geometric objects.

9.2. **History.** But before getting into that, let's first talk about the history of the matching problem. Recall that for a local field F, $\mathcal{A}_G(F) = \mathrm{Rep}(G(F))$ and $\Theta_G(X) = \mathrm{fns}(X(F))$ is some version of representation of functions, and hence is a representation of $G(F)$, i.e., $\Theta_G(X) \in \mathcal{A}_G(F)$. While the study of representations over general local fields is more recent, such a problem (to study how $\Theta_G(X)$ fits in $\mathcal{A}_G(F)$) has been considered for quite a long time.

When $F = \mathbb{R}$, the Lie group G is compact, and X is a compact G-manifold, this dates back to Cartan(1929) and Weyl(1934)[2]. Weyl and his student Peter studied compact Lie group representations via analysis. They proved that $L^2(G)$ can be decomposed into a direct sum of finite dimensional irreducible G-representations by producing compact integral operators on G. Later Cartan proved that for a homogeneous G-space X,

$$L^2(X) \cong \bigoplus_{\pi} \pi^{m(\pi)}, \tag{9.2}$$

where $m(\pi) \in \mathbb{Z}_{\geq 0}$ is the multiplicities of π in $L^2(X)$.

This story is connected to two other things. First, the decomposition (9.2) is closely connected to studying differential operators on X. For example, for $G = \mathrm{SO}_3$ and $X = S^2$, this decomposition is known for representation theorists and analysts. Because it is a decomposition into eigenspaces of the Laplace operator; the decomposition gives spherical harmonic functions. More generally, if D is a differential operator on X commuting with the G-action, then each eigenspace of D in $L^2(X)$ is a G-representation. The most optimal cases happen when all multiplicities are at most one, so that it is easiest for one to find eigenfunctions and eigenvalues[3] (by Schur's Lemma, D will act as a scalar on each π). A lot of generalization later on will have this in mind.

The other thing connected to the decomposition is the study of special functions. Cartan proved $m(\pi) \leq 1$ when X is a compact Riemannian symmetric space. To study how π lives in $L^2(X)$, he constructed the projection of $L^2(X) \to L^2(X)_\pi$ (the isotypic part of π) by constructing a G-invariant kernel on $X \times X$ (equivalently a function on $(X \times X)/G$), which equals a zonal spherical function. This observation is quickly generalized over time. In particular, by the 60s Vilenkin wrote a book[4] on this in a more general setting, where the group G is not necessarily compact and he studied $L^2(X) \to L^2(Y)_\pi$ for two G-spaces X and Y when $L^2(X)_\pi \cong L^2(Y)_\pi \cong \pi$. In this case, the kernel is a function on $(X \times Y)/G$.

[2]Weyl, H. "Harmonics on Homogeneous Manifolds." Annals of Mathematics, vol. 35, no. 3, 1934, pp. 486–99.

[3]Instead of solving differential equations.

[4]Vilenkin, N. Ja. Special functions and the theory of group representations. Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, RI, 1968. x+613 pp.

His observation was by generalizing this approach one can obtain various cases of special functions parametrized by representations, which recovers many of classical special functions, e.g., Bessel functions, hypergeometric functions, and so on in a uniform procedure.

Then Harish-Chandra during 50s to 70s classified representations of a real semisimple group G and wrote down a Plancherel formula for the group case, i.e., he decomposed $L^2(G)$ as a $G \times G$ representation, picked out the basic function δ_e (which is the dirac measure), and realized where δ_e goes in the decomposition. The central goal for Harish-Chandra was the Plancherel measure (i.e. to study the pair $(L^2(G), \delta_e)$), while the classification was obtained in the process of reaching his goal. Starting from 60s to 70s, people started to consider the similar problem for more general spaces. A part of them is analysts who study the Laplace operator on hyperboloids. Most of generalization was done on symmetric varieties.

Around 2010, in Sakalleridis-Venkatesh[5] we proposed that studying the Plancherel measure for X over a local field is a local analogue of studying X-period of automorphic forms. Our book made a connection between harmonic analysis and the problem that people have studied for a long time on automorphic forms. As in previous cases, the multiplicity one assumption[6] plays a special role, which was emphasized by Piatetski-Shapiro. This is the main reason that we have restricted ourselves to consider spherical (more precisely hyperspherical) spaces as they guarantee finite multiplicities and sometimes multiplicity one. While most of BZSV[7] is very algebraic, it is helpful to have L^2 intuition to get correct normalization.

9.3. **Set up.** Suppose one has (9.1) and for simplicity $\check{G}_X = \check{G}$. For instance, we have seen this for $(G, X, \check{M}) = (\mathrm{GL}_1, \mathbb{A}^1, T^*\mathbb{A}^1), (\mathrm{PGL}_2, \mathrm{PGL}_2/\mathbb{G}_m, T^*\mathbb{A}^2)$.

Let V, W be finite dimensional representations of $\check{G}$. Let χ_V, χ_W be characters of V and W respectively, and let T_V, T_W be the associated Hecke operators by the Satake correspondence. We will rewrite both terms in the identity

$$(T_V\delta_0, T_W\delta_0) = \int_{G_c^\vee} \chi_V(g)\overline{\chi}_W(g) \cdot (q\text{-character of } M^\vee)\, d_{\mathrm{Haar}}g$$

in a geometric fashion. The left-hand side will be interpreted as the trace of Frobenius on ?, and the right-hand side will be interpreted as the trace of q-action(=dimension if q were 1) on ??. Here ? and ?? are both hom spaces in different categories. This will motivate our conjecture on the equivalence of categories.

[5]Sakellaridis, Yiannis; Venkatesh, Akshay. Periods and harmonic analysis on spherical varieties. Astérisque No. 396 (2017), viii+360 pp.

[6]The vector space $\mathrm{Hom}_G(C_c^\infty(X(F)), \pi)$ has at most dimension one for any smooth irreducible representation π of $G(F)$.

[7]D. Ben-Zvi, Y. Sakellaridis, A. Venkatesh, Relative Langlands Duality, preprint. Available at https://www.math.ias.edu/ akshay/research/BZSVpaperV1.pdf.

10. Day 3 Lecture 3 (Yiannis)

The first goal of the talk is to associate a map

$$\mathrm{SL}_2 \times \check{G}_X \to \check{G} \tag{10.1}$$

to a spherical G-variety X. This map is motivated from the study of the $\mathcal{H}_G$-module $C_c^\infty(X(F))^{G(\mathfrak{o})}$, which we expects, for 'good' X, an isomorphism

$$C_c^\infty(X(F))^{G(\mathfrak{o})} \cong \mathbb{C}[\mathrm{Rep}\ \check{G}_X], \tag{10.2}$$

of modules over $\mathcal{H}_G \cong \mathbb{C}[\mathrm{Rep}\ \check{G}]$. The map (10.1) is used to described the right-handed side of this isomorphism. Before explaining the right-handed side, we will first define the dual group $\check{G}_X \hookrightarrow \check{G}$ associated to X by defining its Cartan subgroup A_X and its Weyl group W_X.

10.1. **Relative Cartan subgroup.** Let B be a Borel subgroup of G, X° be the corresponding open B-orbit.

Associated to G, one can define a torus A, called the universal Cartan of G, such that for any Borel subgroup B together with its unipotent radical N, there is a canonical isomorphism $B/N \xrightarrow{\sim} A$. As B acts on X°, A acts on X°/N and the action factors through a quotient A_X of A.

The character group $X^*(A_X)$ of A_X can be associated with the group of B-eigencharacters appearing in the B-module $k(X)$. Indeed, as X has an open dense B-orbit, the B-eigencharacters $\chi \in X^*(B)$ of $k(X)$ are determined by their associated nonzero B-eigenfunctions $f_\chi \in k(X)^{(B)}$, which is unique up to scalar by k (i.e. the short exact sequence (10.3)). We also have

$$k(X)^{(B)} = k(X^\circ)^{(B)} = k(B_x\backslash B)^{(B)} = k(B_x\backslash B/N)^{(B/N)} = k(X^\circ/N)^{(A)},$$

where the B-orbit $X^\circ \cong B_x\backslash B$ for $x \in X^\circ$, B_x the corresponding stablizer subgroup.

Under the association of $X^*(A_X)$ as the group of B-eigencharacters of $k(X)$, we have a short exact sequence

$$1 \to k^* \to k(X)^{(B)} \xrightarrow{f_\chi \mapsto \chi} X^*(A_X) \to 1 \tag{10.3}$$

10.2. **Little Weyl group.** There are three equivalent ways to define/see the little Weyl group W_X for a spherical G-variety X. The last one is the most computable.

10.2.1. *From G-invariant valuations.* Let $\mathcal{V}$ be the set of G-invariant $\mathbb{Q}$-valued discrete valuations on $k(X)$. Using the exact sequence (10.3), Knop showed that we have an injection map

$$\begin{aligned} \mathcal{V} &\hookrightarrow \mathrm{Hom}(X^*(A_X), \mathbb{Q}) = X_*(A_X) \otimes \mathbb{Q} =: \mathfrak{a}_X \\ v &\mapsto (f_\chi \mapsto v(f_\chi)) \end{aligned}$$

Proposition 10.4 (Knop, The Luna-Vust theory of Spherical Embeddings)**.** *There is a subgroup $W_X \subset W$ that stabilizes $X^*(A_X) \subset X^*(A)$ and such that $\mathcal{V}$ is a fundamental domain for the action of W_X on $\mathfrak{a}_X$.*

10.2.2. *From GIT quotient of cotangent bundle.*

Proposition 10.5 (Knop)**.** *For homogeneous, quasi-affine spherical G-variety X, let $\mathfrak{a}_X = \mathrm{Lie}\ A_X, \mathfrak{a} = \mathrm{Lie}\ A$ then $\mathfrak{a}_X^* \subset \mathfrak{a}^*$. There is a subgroup $W_X \subset W$ that stabilizes $\mathfrak{a}_X^*$, and a canonical isomorphism*

$$T^*X /\!/ G \xrightarrow{\sim} \mathfrak{a}_X^* /\!/ W_X.$$

Furthermore, the map $T^*X \to \mathfrak{g}^* \to \mathfrak{a}^* /\!/ W$ factors through the above map.

Example 10.6 (Group case)**.** When $X = H, G = H \times H$ acting on X on the right by $h \cdot (h_1, h_2) = h_1^{-1} h h_2$. Then $T^*H /\!/ G = T_e^*H /\!/ \mathrm{Stab}_G(e) = \mathfrak{h}^* /\!/ H$ and by Chevalley's isomorphism $\mathfrak{h}^* /\!/ H = \mathfrak{a}_H^* /\!/ W_H$, where $\mathfrak{a}_H$ is the Lie algebra of the maximal torus in H, and W_H is the Weyl group of H. Thus, we find W_H is the little Weyl group of X.

10.2.3. *From Knop's Weyl group action on the Borel orbits.* Knop in his paper 'On the set of orbits for a Borel subgroup' defined the following action of W_G on the B-orbits of X:

Definition 10.7 (Action of W_G on B-orbits of X)**.** Let $\alpha \in \Delta_G$ be a simple root of G with the corresponding reflection $w_\alpha \in W_G$. Let γ be a B-orbit in X, $P_\alpha \supset B$ be the minimal parabolic corresponding to α, then $w_\alpha \cdot \gamma$ will be a B-orbit in γP_α, defined as follows: $\gamma P_\alpha / R(P_\alpha)$ is a homogeneous spherical $P_\alpha / R(P_\alpha) \cong \mathrm{PGL}_2$-variety. Up to equivalence there are only 4 types of homogeneous spherical PGL_2-variety, and we define the action of w_α on the Borel orbits of each in the following table:

Type	Homogeneous spherical PGL_2-variety	Borel orbits and action of w_α
G	pt	one closed orbit
T	$\mathbb{G}_m \backslash \mathrm{PGL}_2$	two closed orbits and one open one; w_α permutes the two closed orbits and fixes the open one
N	$N(\mathbb{G}_m) \backslash \mathrm{PGL}_2$	one closed and one open orbit; w_α fixes the orbits
U	$N \cdot S \backslash \mathrm{PGL}_2$ for $N = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \subset S \subset B$	one open and open closed orbit; w_α permutes the two orbits

Together, this defines an action of the Weyl group W_G of G on the B-orbits of X.

Definition 10.8. Given a spherical G-variety X, one can associate a parabolic subgroup $P(X) := \{g \in G | X^\circ g = X^\circ\}$ together with its Levi subgroup $L(X)$. We say X is *tempered* if $P(X) = B$.

Example 10.9 (Temperedness)**.** The group case is tempered.

The little Weyl group appears under Knop's action as follows:

Proposition 10.10 (Knop)**.** *Under this action, the the stablizer of the open B-orbit X° is*

$$W_X \ltimes W_{P(X)},$$

where $W_{P(X)}$ is the Weyl group of $P(X)$.

We will use this proposition to compute little Weyl's groups in some cases of X.

Example 10.11 (Group case)**.** Let $X = H$ and $G = H \times H$ acting on X on the right by $h \cdot (h_1, h_2) = h_1^{-1} h h_2$. Another way to view X is as $X = H^{\mathrm{diag}} \backslash H \times H$ where $G = H \times H$ acts by right-multiplication.

By Bruhat decomposition, the $B \times B$-orbits of X are BwB over $w \in W_H$ where W_H is the Weyl group of H. The open $B \times B$-orbit is Bw_0B with $w_0 \in W_H$ being the longest element defined by $B \subset H$.

For a simple root $(\alpha, 0) \in \Delta_{H\times H} = \Delta_H \sqcup \Delta_H$ of $H \times H$, the corresponding minimal parabolic is $P_\alpha \times B$, we consider $(Bw_0B) \cdot (P_\alpha \times B)/R(P_\alpha \times B) = R(P_\alpha)\backslash P_\alpha w_0 B/B$. This is always type U [8] (see the above table). In particular, we know what two Borel orbits that $w_{(\alpha,0)} = (w_\alpha, 1) \in W_{H\times H} = W_H \times W_H$ permutes in $P_\alpha w_0 B$. Indeed, as $P_\alpha = B \sqcup Bw_\alpha B$ and $w_0(\alpha) \in \Phi^-$ so [9]

$$P_\alpha w_0 B = Bw_0 B \sqcup Bw_\alpha w_0 B.$$

Similarly, $w_{(0,\alpha)} = (1, w_\alpha) \in W_{H\times H} = W_H \times W_H$ permutes Bw_0B and $Bw_0w_\alpha B = Bw_\alpha w_0 B$ (as $ww_\alpha w^{-1} = w_{w(\alpha)}$ for any $w \in W, \alpha \in \Phi$). This implies (w_α, w_α) fixes the open Borel orbit $X^\circ = Bw_0B$. Furthermore, $(w, w') \in W_{H\times H}$ fixes X° if and only if $w = w'$.

As X is tempered so $W_{P(X)} = 1$, we obtain $W_X = \{(w, w) : w \in W_H\}$ from Knop's action.

Example 10.12 ($X = \mathrm{PGL}_2^{\mathrm{diag}} \backslash \mathrm{PGL}_2^3$ as PGL_2^3-space.)**.** Let B denote the upper triangular matrices in PGL_2. First we compute the open Borel orbit of X, i.e., the largest double coset in $PGL_2^{diag} \backslash PGL_2^3/B^3$. Identify PGL_2^3/B^3 with $(\mathbb{P}^1)^3$ and one can see that the largest orbit corresponds to the representative $x = ([1,0],[0,1],[1,1]) \in (\mathbb{P}^1)^3$: since PGL_2 acts transitively on $\mathbb{P}^1$, to get a large orbit one needs three distinct points.

First we compute A_X by computing the stabilizer B_x. The identification PGL_2/B with $\mathbb{P}^1$ is given by

$$\begin{pmatrix} a & * \\ c & * \end{pmatrix} \leftrightsquigarrow [a, c].$$

Therefore, after working out some matrix multiplications (note that here B^3 acts on the right), one see that $B_x = \{1\}$ and hence $A_X = \mathbb{G}_m^3$.

[8] One can show the stablizer of $R(P_\alpha)w_0B \in R(P_\alpha)\backslash P_\alpha w_0 B/B$ under the action of $R(P_\alpha)\backslash P_\alpha$ is $R(P_\alpha)\backslash B^- \cap P_\alpha$, hence we have isomorphisms $\mathrm{PGL}_2/B^- \cong R(P_\alpha)\backslash P_\alpha/(B^- \cap P_\alpha) \xrightarrow{[g]\mapsto[gw_0]} R(P_\alpha)\backslash P_\alpha w_0 B/B$

[9] see the proof of Bruhat decomposition from Tits systems

Now we compute W_X. For $(1,1,w)$, where w is the nontrivial element in W_{PGL_2}, the corresponding minimal parabolic is $B \times B \times PGL_2$. Pick a representative of x in PGL_2^3 and compute the quotient

$$\begin{aligned} Y_2 &= PGL_2^{diag}\backslash \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} B \right) \cdot B \times B \times PGL_2/B \times B \times \{1\} \\ &= PGL_2^{diag}\backslash B \times wB \times PGL_2/B \times B \times \{1\} \\ &\cong PGL_2\backslash PGL_2 \\ &= \text{point}. \end{aligned}$$

Therefore, $(1,1,w)$ stabilizes the open Borel orbit. The situation for the other two simple roots are similar. One can check that the parabolic associated to X is B^3 and thus conclude that $W_X = W_{PGL_2^3}$.

Example 10.13 (Other examples in the exercises)**.** (1) $X = \mathrm{Mat}_n$ as $\mathrm{GL}_n \times \mathrm{GL}_n$-space.
(2) $X = \mathrm{SO}_n\backslash \mathrm{SO}_{n+1}$ as SO_{n+1}-space.
(3) $X = \mathrm{SO}_{n+1}$ as $\mathrm{SO}_n \times \mathrm{SO}_{n+1}$-space. Expected answer: $\check{G}_X = \check{G}$.

10.3. **Dual group of a spherical variety.** The dual group $\check{G}_X$, viewed as a subgroup of $\check{G}$, is determined from knowing its Cartan subgroup A_X and its Weyl group W_X.

Example 10.14 (Group case)**.** When $X = H$ and $G = H \times H$ then we show $\check{G}_X = \check{H}$

Example 10.15. When $X = \mathrm{PGL}_2^{\mathrm{diag}}\backslash \mathrm{PGL}_2^3$ then $\check{G}_X = \check{G}$

Example 10.16 (Hecke)**.** When $X = \mathbb{G}_m\backslash \mathrm{PGL}_2$

10.4. **The SL_2-term.** See BZSV. §4

We have defined $i : \check{G}_X \to \check{G}$. With this, $\mathbb{C}[\mathrm{Rep}\ \check{G}_X]$ is a module over $\mathbb{C}[\mathrm{Rep}\ \check{G}]$, where $V \in \mathrm{Rep}\ \check{G}$ acts on $W \in \mathrm{Rep}\ \check{G}_X$ by $i^*V \otimes W$ where $i : \check{G}_X \to \check{G}$. However, this is not the correct module structure giving rise to (10.2). A shift in i is needed, illustrated in the below example.

Example 10.17 (Whittaker)**.** Let $X = pt = G\backslash G$. $\mathcal{H}_G$ acts on $C_c^\infty(X(F))^{G(\mathfrak{o})} = \mathbb{C}$ through the trivial character $\chi_1 : \mathcal{H}_G \to \mathbb{C}$. Under Satake's isomorphism $\mathrm{Hom}(\mathcal{H}_G, \mathbb{C}) = (\check{A} /\!\!/ W)(\mathbb{C})$, χ_1 corresponds to $q^\rho \in \check{A}(\mathbb{C})$.

How to interpret this q^ρ: $\rho \in \check{\mathfrak{g}}$ corresponds to $\rho = de^\rho$ where $e^\rho : \mathbb{G}_m \to \check{G}$, and $q^\rho = e^\rho(q)$.

One can associate to X a $\mathfrak{sl}_2$-triple (h,e,f) that is principal in $\check{L}(X) \subset \check{G}$, the Levi subgroup corresponding to $P(X) \subset G$. This triple commutes with $\check{G}_X$, defining $\iota : q^{h/2}\check{G}_X \hookrightarrow \check{G}$, which lifts to $\mathrm{SL}_2 \times \check{G}_X \to \check{G}$. This shift ι of i gives us (10.2).

Example 10.18 (Whittaker)**.** Check that the shifts is correct with respect to the action. The $\mathfrak{sl}_2$ triple (h,e,f) is principal, where $P(X)$

10.5. **Hyperspherical Varieties.** The second goal of the talk is to generalize spherical varieties. This comes the notion of *hyperspherical variety.* The basic example is the cotangent space of a spherical variety.

Definition 10.19. A Hamiltonian G-variety M with a commuting $\mathbb{G}_{gr}$ is *hyperspherical* if it satisfies the following conditions:

(1) M is smooth and affine.
(2) M is coisotropic as a G-variety.
(3) The generic stabilizers of the G-action on M are connected.
(4) The image of the moment map $M \to \mathfrak{g}^*$ meets the nilpotent cone (i.e., has nontrivial intersection).
(5) The grading action is "neutral". In particular, it is compatible with the squaring action on $\mathfrak{g}^*$.

Remark 10.20.
- For (4), If $M = T^*X$, then the image of the moment map always contains the zero section.
- Experiments show that (2) + (3) is dual to (1).

Example 10.21 (When M is not of the form T^*X)**.** Consider $T^*(N\backslash G)$, where N is the unipotent radical of B. We interpret the cotangent space of the quotient via *Hamiltonian reduction*, which is defined to be the quotient of the fiber of $\{0\}$ under the moment map by N:

$$T^*(N\backslash G) \overset{def}{=} N \mathbin{/\mkern-6mu/\mkern-6mu/} G = \{0\} \times^N_{\mathfrak{n}^*} T^*G.$$

That is, we are looking at cotangent vectors on G that are normal to the N-orbits. Notice that this space is *not* affine.

A slight modification is the *Whittaker cotangent space.* To construct it, first take (h, e, f) an $\mathfrak{sl}_2$-triple in $\mathfrak{g} \cong \mathfrak{g}^*$. We want f to be regular nilpotent in $\mathfrak{g}^*$ so that f gives a nondegenerate character of $\mathfrak{n}$. We put the Whittaker cotangent space to be

$$T^*((N, f)\backslash G) = \{f\} \times^N_{\mathfrak{n}^*} T^*G = M.$$

In this way, M is affine and hyperspherical. Note that it also comes equipped with a Kostant section

$$M \mathbin{/\mkern-6mu/} G = \mathfrak{a}^* \mathbin{/\mkern-6mu/} W \xrightarrow{Kostant} M.$$

This gives $M \cong (\mathfrak{a}^* \mathbin{/\mkern-6mu/} W) \times G$

Theorem 10.22 (Structure theorem of hyperspherical varieties)**.** *The hyperspherical Hamiltonian G-variety M is built out of the following data:*

$$(G, H, \iota, S),$$

where G is the reductive group G, H is a reductive subgroup of G, ι is a $\mathfrak{sl}_2$-triple in G, and S a symplectic representation of G.

The above construction is called the *Whittaker indcution*, $M = \mathrm{W}_\iota\mathrm{Ind}_H^G(S)$.

For such space M, we expect some dual space $\check{M}$.

In the case of $M = T^*X$, X spherical, then we have the correspondence

$$\check{M} \longleftrightarrow (\check{G}, \check{G}_X, \iota_{P(X)}, \check{S})$$

where $\check{G}$ is the dual group, $\check{G}_X$ is the reductive subgroup associated to X defined earlier, $\iota_{P(X)}$ is the $\mathfrak{sl}_2$-triple corresponding to $P(X)$, and $\check{S}$ is a symplectic $\check{G}$-representation determined (up to isomorphism) by the *colors* of X, i.e., by the B-stable divisors that are not G-stable.

Noncanonically, we have

$$\check{M} \cong V_X \times^{\check{G}_X} \check{G}, \qquad V_X = \check{S} \oplus (\check{\mathfrak{g}}/\check{\mathfrak{g}}_X)_e + \check{\mathfrak{g}}_1,$$

where e comes from the $\mathfrak{sl}_2$-triple $\iota_{P(X)}$ and $\check{\mathfrak{g}}_1$ is the degree 1 part in $\check{\mathfrak{g}}$.

For an arbitrary $\mathfrak{sl}_2$-triple $\iota\colon \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$, h induces a grading of $\mathfrak{g}$ by the eigenvalues of $[h, -]$, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$, where we always have that $f \in \mathfrak{g}_{-2}$. If this $\mathfrak{sl}_2$-triple gives an even decomposition (i.e., only has even degrees), then under the identification $\mathfrak{g} \cong \mathfrak{g}^*$, f defines a character $f\colon \mathfrak{g}_+ \to F$. In the general case, f defines a character $\mathfrak{g}_{\geq 2} \times^{\mathfrak{g}_{\geq 2}} F$ (the Heisenberg group).

11. Day 4 Lecture 1, Akshay

11.1. **Plancherel formula.** Recall that we have two TQFT's $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$. Taking $\mathcal{Y} := \overline{\mathbb{F}}_q((t))$, we expect an equivalence of 2-categories: $\mathcal{A}_G(\mathcal{Y}) \to \mathcal{B}_{G^\vee}(\mathcal{Y})$. Given the extra data of the spherical G-variety X, we have associated boundary theories $\Theta_{G,X} \in \mathcal{A}_G(\mathcal{Y})$ and $\Theta_{G^\vee,X^\vee} \in \mathcal{B}_{G^\vee}(\mathcal{Y})$. We expect these two 1-categories to be equivalent as well.

We first analyze the decategorified version of the equivalence we want. Take $F = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$. Assume X is a spherical G-variety. Under certain finiteness assumptions, we see

$$\mathcal{C}_c(X(F))^{G(\mathcal{O})} \cong \mathbb{C}[\mathrm{Rep}\ G_X^\vee] \tag{11.1}$$

for some group $G_X^\vee \subset G^\vee$. For the purpose of this talk, we assume $G_X^\vee = G^\vee$ (this is the so-called strongly tempered condition in BZSV). The above isomorphism is not only an isomorphism of modules of $\mathcal{H}(G(F), G(\mathcal{O})) \cong \mathbb{C}[\mathrm{Rep}\ G^\vee]$, but it also preserves inner product. Let $f, g \in \mathcal{C}_c(X(F))^{G(\mathcal{O})}$, and $\hat{f}, \hat{g}$ be their images under (11.1), then we have

$$\langle f, g\rangle = \int_{G_c^\vee} \langle \hat{f}, \hat{g}\rangle \cdot q\text{-char}(G^\vee \times \mathbb{G}_m^{gr}, M^\vee) d\mu_H. \tag{11.2}$$

Here are what the notations mean:

- $G_c^\vee$ is a compact form of the complex reductive group $G^\vee$.
- $M^\vee$ is some space with an action of $G^\vee \times \mathbb{G}_m^{gr}$. Usually it is $T^*X^\vee$.
- The q-character on $M^\vee$ means the q-character of $\mathbb{C}[M^\vee]$ as a representation of $G^\vee \times \mathbb{G}_m^{gr}$.
- μ_H is a Haar measure.

Under the isomorphism (11.1), the characteristic function $\mathbb{1}_{X(\mathcal{O})}$ should go to the trivial representation.

Example 11.3. We have computed Plancherel formulas for the following two cases:

(1) $G = \mathbb{G}_m$, $X = \mathbb{A}^1$. In this case, $G^\vee = \mathbb{G}_m$ and $M^\vee = T^*\mathbb{A}^1$.

(2) $G = PGL_2$, $X = PGL_2/\mathbb{G}_m$. In this case $G^\vee = SL_2$ and $M^\vee = T^*\mathbb{A}^2$.

11.2. **Inner product as trace on Hom spaces.** One key observation is that both sides of (11.2) can be realized as trace of certain operators on some Hom spaces. We assume $\hat{f} = \chi_V$ (the character of representation V) and $\hat{g} = \chi_W$. In other words, $f = h_V * \mathbb{1}_{X(\mathcal{O})}$ and $g = h_W * \mathbb{1}_{X(\mathcal{O})}$. For simplicity, we call LHS the left hand side of (11.2) and RHS the right hand side of (11.2).

Notice that if A, B are complex representations of $G^\vee$, then $\chi_A\overline{\chi}_B$ is the character of $\mathrm{Hom}(B, A)$, so

$$\int_{G_c^\vee} \chi_A\overline{\chi}_B d\mu_H = \dim \mathrm{Hom}(B, A)^{G^\vee} = \dim \mathrm{Hom}_{G^\vee}(B, A). \tag{11.4}$$

So

$$\mathrm{RHS} = \int_{G_c^\vee} \chi_V\overline{\chi}_W(\sum_i q^{-\frac{i}{2}}\chi_{\mathbb{C}[M^\vee]_i})d\mu_H \tag{11.5}$$

$$= \sum_i q^{-\frac{i}{2}} \dim \mathrm{Hom}_{G^\vee}(W, V \otimes \mathbb{C}[M^\vee]_i)d\mu_H \tag{11.6}$$

$$= \mathrm{trace}(q^{-\frac{1}{2}}, \mathrm{Hom}_{G^\vee}(W, V \otimes \mathbb{C}[M^\vee])). \tag{11.7}$$

In the equality, we are viewing $q^{-\frac{1}{2}}$ as an element in $\mathbb{G}_m^{gr}(\mathbb{C}) = \mathbb{C}^*$.

Now we can further get a more symmetric presentation by viewing a $\mathbb{C}$-linear $W \to V \otimes \mathbb{C}[M^\vee]$ as a $\mathbb{C}[M^\vee]$-linear map $W \otimes \mathbb{C}[M^\vee] \to V \otimes \mathbb{C}[M^\vee]$. So

$$\mathrm{RHS} = \mathrm{trace}(q^{-\frac{1}{2}}, \mathrm{Hom}_{G^\vee, \mathbb{C}[M^\vee]}(W \otimes \mathbb{C}[M^\vee], V \otimes \mathbb{C}[M^\vee])). \tag{11.8}$$

Let $\underline{V} = V \otimes \mathbb{C}[M^\vee]$. Then it is a $\mathcal{O}(M^\vee)$ module with a $G^\vee$ action. So we view it as a coherent sheaf on $M^\vee/G^\vee$. Similarly we define $\underline{W}$. Then

$$\mathrm{RHS} = \mathrm{trace}(q^{-\frac{1}{2}}, \mathrm{Hom}_{M^\vee/G^\vee}(\underline{W}, \underline{V})). \tag{11.9}$$

The Hom is taken in the derived category of quasi-coherent (indcoherent?) sheaves on $M^\vee/G^\vee$.

The left hand side of (11.2) can be realized as a trace using the sheaf-function correspondence. Recall the following general fact: suppose Z is a variety defined over $\mathbb{F}_q$, and $\mathcal{F}, \mathcal{G}$ are mixed $\overline{\mathbb{Q}}_l$-sheaves on $Z_{\overline{\mathbb{F}}_q}$, then taking trace of Frobenius one obtains functions f, g on $Z(\mathbb{F}_q)$. We have

$$\mathrm{trace}(\mathrm{Fr}^{-1} | \mathrm{Hom}_{Z_{\overline{\mathbb{F}}_q}}(\mathcal{F}, \mathcal{G})) = \sum_{z \in Z(\mathbb{F}_q)} f(z)\overline{g}(z). \tag{11.10}$$

Note that $\overline{g}$ is the trace of $\mathbb{D}\mathcal{G}$ (the Verdier dual of $\mathcal{G}$). For example, if we take $\mathcal{F} = \mathbb{D}\mathcal{G} = \underline{\overline{\mathbb{Q}}_l}_Z$, the constant sheaf on Z, then the above reduces to the Lefschetz fixed-point formula.

Now let $F = \mathbb{F}_q((t))$ and $\mathcal{O} = \mathbb{F}_q[[t]]$. For many nice X defined over $\mathbb{F}_q$, $X(F)/G(\mathcal{O})$ is the $\mathbb{F}_q$ point of some "reasonable algebraic stack" $X_F/G_{\mathcal{O}}$. For example, when $X = H$ and $G = H \times H$, $X(F)/G(\mathcal{O})$ is the $\mathbb{F}_q$ points of the affine Grassmanian Gr_G, which is an ind-scheme.

The Hecke algebra $\mathcal{H}_G$ action on $\mathcal{C}_c(X(F))^{G(\mathcal{O})}$ can be upgraded to an action of the Hecke category on sheaves on $X_F/G_{\mathcal{O}}$. The Hecke category here is the derived category of mixed $\overline{\mathbb{Q}}_\ell$-sheaves $G(\mathcal{O})$-equivariant constructible sheaves on Gr_G. Let IC_V denote the perverse sheaf corresponding to representation V under the geometric Satake isomorphism. Let $\delta_0 := j_*(\underline{\overline{\mathbb{Q}}_l}_{X_{\mathcal{O}}/G_{\mathcal{O}}})$, where $j : X_{\mathcal{O}}/G_{\mathcal{O}} \to X_F/G_{\mathcal{O}}$ is a closed embedding (?). Then f, g in LHS are the functions associated to $IC_W * \delta_0$ and $IC_V * \delta_0$, respectively, under the sheaf-function correspondence.

So we can write

$$\text{LHS} = \mathrm{trace}(\mathrm{Fr}^{-1} | \mathrm{Hom}(IC_W * \delta_0, IC_V * \delta_0)). \tag{11.11}$$

Here Hom is evaluated in the category of constructible sheaves on $X_F/G_{\mathcal{O}}$.

11.3. **The local conjecture.** The discussion in the previous section strongly suggests an equivalence of categories (not true, to be corrected):

$$\mathrm{Sh}_c(X_F/G_{\mathcal{O}}) \cong QCoh(M^\vee/G^\vee). \tag{11.12}$$

The equivalence should satisfy the following:

- δ_0 should be sent to the structure sheaf $\mathcal{O}$ on $M^\vee/G^\vee$.
- Frobenius action on the left should correspond to the action of $q^{\frac{1}{2}} \in \mathbb{G}_m^{gr}$ on the right.
- The equivalence is compatible with the module structure of the Hecke category, under the derived Satake isomorphism. On the level of abelian categories, $\mathcal{H}_G$ acts on the left by convolution and Rep $(G^\vee)$ acts on the right by tensor product: for any representation V of $G^\vee$, one can pullback along $M^\vee/G^\vee \to */G^\vee$ to get a vector bundle on $M^\vee/G^\vee$. The action of V is by tensoring with this vector bundle.

This equivalence can be seen as matching the boundary conditions as discussed in the beginning of Section (11.1).

The equivalence above is not correct as stated. Before we mention the correction, let us do some plausibility check. If the equivalence if correct, we expect the following isomorphism:

$$\mathrm{Hom}_{\mathrm{Sh}_c(X_F/G_{\mathcal{O}})}(\delta_0, \delta_0) \cong \mathrm{Hom}_{M^\vee/G^\vee}(\mathcal{O}, \mathcal{O}). \tag{11.13}$$

The left hand side equals $H^*(X_{\mathcal{O}}/G_{\mathcal{O}})$. Notice that $G_{\mathcal{O}}/G$ is pro-unipotent and therefore contractible, it doesn't contribute to the cohomology. Similarly $X_{\mathcal{O}}$ contracts to $X(\overline{\mathbb{F}}_q)$. We have

$$H^*(X_{\mathcal{O}}/G_{\mathcal{O}}) \cong H^*_G(X(\mathbb{C})). \tag{11.14}$$

On the right hand side, we have

$$\mathrm{Hom}_{QCoh(M^\vee/G^\vee)}(\mathcal{O},\mathcal{O}) \cong \mathbb{C}[M^\vee]^{G^\vee}. \tag{11.15}$$

So we should check:

$$H^*_G(X(\mathbb{C})) = \mathbb{C}[M^\vee]^{G^\vee}. \tag{11.16}$$

Example 11.17. Consider the case $G = \mathbb{G}_m$, $X = \mathbb{A}^1$, $G^\vee = \mathbb{G}_m$ and $M^\vee = T^*\mathbb{A}^1$. Then $H^*_G(X) = \mathbb{C}[\xi_2]$, where ξ_2 lies in cohomological degree 2. Recall $\lambda \in G^\vee$ acts on $(x, y) \in T^*\mathbb{A}^1$ by $(\lambda x, \lambda^{-1}y)$. We have $\mathbb{C}[M^\vee]^{G^\vee} = \mathbb{C}[xy]$

Example 11.18. Consider the case $G = PGL_2, X = PGL_2/\mathbb{G}_m$, $G^\vee = SL_2$ and $M^\vee = T^*\mathbb{A}^2$. Then $H^*_G(X) = H^*_{\mathbb{G}_m}(\mathrm{pt}) = \mathbb{C}[\xi_2]$.

On the other hand, realizing $M^\vee = T^*\mathbb{A}^2$, and SL_2 acts by left multiplication, we have $\mathbb{C}[M^\vee]^{G^\vee} = \mathbb{C}[\det]$.

In both examples we have checked the algebras are isomorphic, but we have to assign the correct grading on the right hand side. On the categorical level, there is a procedure called *shearing* and the corrected conjecture is the following:

$$\mathrm{Sh}_c(X_F/G_{\mathcal{O}}) \cong \mathrm{IndCoh}(M^\vee/G^\vee)^\natural. \tag{11.19}$$

12. Day 4 Lecture 2

In this light lecture, we will switch to the global setting and give some motivation and history. We will begin by finishing one thing that we didn't get to in the previous lecture. By following the reasoning we went through, one can even obtain a general unramified Plancherel formula.

Let F be a local field. The "package" $(C^\infty_c(X_F)^{G_O}, \delta_e, \langle -,-\rangle)$ can then be described, i.e., the Hecke action on it can be diagonalized, completely in terms of the dual $\check{M}$. We will describe the answer under a slight simplifying assumption, namely in the case when the $\mathbb{G}_m$-action on $\check{M}$, which is always of the form $\check{G} \times_{\check{G}_X} V_X$ for some $\check{G}_X$-representation V_X, comes entirely from an action on V_X alone. Then, there is an isomorphism

$$C^\infty_c(X_F)^{G_O} \xrightarrow{\cong} \mathbb{C}[\check{G}_X]^{\check{G}_X}$$

which sends δ_e to the constant function 1, and which sends the inner product $\langle -,-\rangle$ to the integral

$$\langle -,-\rangle \mapsto \int_{\check{G}_X^{\mathrm{compact}}} \langle -,-\rangle \cdot (q\text{-character of } V_X) \cdot d\mathrm{Haar}.$$

Recall that the product of the Haar measure with the q-character of V_X is called the Plancherel measure.

That's all for the local setting; now we will go to the global setting. Our theories $\mathcal{A}_G$ and $\mathcal{B}_{\check{G}}$ will be evaluated on an arithmetic manifold, where $\mathcal{Y}$ is either a curve over $\mathbb{F}_q$ or a curve over the algebraic closure $\overline{\mathbb{F}_q}$. In the first case, $\mathcal{A}_G(\mathcal{Y})$ and $\mathcal{B}_{\check{G}}(\mathcal{Y})$ are both vector spaces; and in the second case, both are categories. The data of $G \circlearrowleft X$ and $\check{G} \circlearrowleft \check{X}$ (really, one should think of these as the data of M and $\check{M}$), define objects $\Theta_{G,X} \in \mathcal{A}_G(\mathcal{Y})$ and $\Theta_{\check{G},\check{X}} \in \mathcal{B}_{\check{G}}(\mathcal{Y})$. In this lecture, we will focus only on defining $\Theta_{G,X} \in \mathcal{A}_G(\mathcal{Y})$. (Everything works in the number field case too, but I will stick to the function field case for ease.)

We have a curve Σ over $\mathbb{F}_q$, and the vector space $\mathcal{A}_G(\Sigma)$ is the vector space of functions on the set $[G]$ of G-bundles over Σ. These G-bundles are defined over $\mathbb{F}_q$. One can understand the set $[G]$ of G-bundles over Σ either adelically, or geometrically. If K is the function field of Σ, then $[G]$ can be written as $G(K)\backslash G(\mathbb{A})/G(\hat{\mathcal{O}})$; a point of $[G]$ will be denoted by $[g]$, and for each closed point $s \in \Sigma$, we will write $g_s \in G(\hat{\mathcal{O}}_s)$ to denote the corresponding element.

The *period function* $\Theta_{G,X} \in \mathcal{A}_G(\Sigma)$ can be defined in three different ways.

- Geometrically: $\Theta_{G,X}([g])$ is the number of sections of the X-bundle associated to $[g]$. Recall that this X-bundle is just $(X \times [g])/G$.
- $\Theta_{G,X}([g])$ is the number of sections $x \in X(K)$ such that for all closed points $s \in \Sigma$, the section xg_s belongs to $X(\hat{\mathcal{O}}_s)$. To see that this is the same as the number from the first bullet, note that if you restrict this X-bundle to the generic point of the curve, it is just X (because the bundle is locally trivial); so a section of the X-bundle associated to $[g]$ is, generically, just a point $x \in X(K)$. Asking that this section is globally defined is equivalent to asking that x not have any poles at any closed point of Σ, i.e., that it lie in $X(\hat{\mathcal{O}}_s)$.
- Arithmetically: $\Theta_{G,X}([g])$ is the sum $\sum_{x \in X(K)} \Phi(x \cdot g)$, where Φ is the characteristic function of $\prod X(\hat{\mathcal{O}}_s)$ inside $X(\mathbb{A})$.

Let's work out some examples of this. Take Σ to be $\mathbb{P}^1$, and G to be $\mathbb{G}_m$. In this case, all $\mathbb{G}_m$-bundles on $\mathbb{P}^1$ are just specified by an integer (namely, n corresponds to $\mathcal{O}(n)$), so $[G] = \mathbb{Z}$. Now, let us take $X = \mathbb{A}^1$. Taking the associated X-bundle just amounts to taking the total space of the line bundle, so that $\Theta_{G,X} =: \Theta_X$ just sends n to the number of global sections of $\mathcal{O}(n)$. In other words:

$$\Theta_X(n) = \begin{cases} q^{n+1} & n \geq 0, \\ 1 & n \leq -1. \end{cases}$$

One important remark is that in our paper, we need to normalize the period function correctly. So we in fact use the slight variant of the period function which, in this case, multiplies the above function Θ_X by a factor of $q^{-n/2}$. So, this rescaling sends

$$n \mapsto \begin{cases} q^{n/2+1} & n \geq 0, \\ q^{-n/2} & n \leq -1. \end{cases}$$

Observe that now, it is symmetric about $n = -1$. (This normalization of the period function is motivated by L^2-theory.)

Let us do one more example. Suppose we take a subgroup $H \subseteq G$. In this case, we can take $X = G/H$, and consider the "pairing" of $\Theta_X := \Theta_{G,X}$ with some function $f : [G] \to \mathbb{C}$. (When $G = \mathrm{PGL}_2$ and $H = \mathbb{G}_m$, this pairing was computed by Hecke in the 1930s.) By the "pairing", I mean:

- In terms of the adelic uniformization, the integral $\int_{G(K)\backslash G(\mathbb{A})} \Theta_X(g) f(g) dg$. Here, I choose the Haar measure to make $G(\hat{\mathcal{O}})$ have volume 1.
- In terms of the geometric picture, the sum $\sum_{[g]\in[G]} \frac{1}{\#\mathrm{Aut}([g])} \Theta_X(g) f(g)$. For instance, in the $\mathbb{G}_m$-case, the measure of the point $[\mathcal{O}(n)]$ is $1/\#\mathbb{G}_m(\mathbb{F}_q) = 1/(q-1)$.

Both of these give you the same number. A good exercise is to check that, when $X = G/H$, either of these pairings can be rewritten in terms of H. Namely:

$$\sum_{[g]\in[G]} \frac{1}{\#\mathrm{Aut}([g])} \Theta_X(g) f(g) = \sum_{[h]\in[H]} \frac{1}{\#\mathrm{Aut}([h])} f(h),$$

$$\int_{G(K)\backslash G(\mathbb{A})} \Theta_X(g) f(g) dg = \int_{[H]} f(h) dh,$$

and all of these numbers are equal, and give you $\langle \Theta_X, f \rangle$. This is a helpful exercise.

Now, we will look at the history of the study of such pairings $\langle \Theta_X, f \rangle$, and why number-theorists are interested in them to begin with. For this, let us go all the way back to the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. This has three properties, which are collectively extremely rigid:

- It has an Euler product $\prod_p (1 - p^{-s})^{-1}$.
- It has an analytic (meaning meromorphic) continuation.
- It has a functional equation relating the values at s and $1 - s$.

It is not so easy for a function to have all three such properties; we will just refer to such as "L-functions". People started to discover such L-functions, and all of them had to do with arithmetic. There were functions associated to Dirichlet characters, number fields, Galois representations, and later elliptic curves. But it was hard to prove that these functions had all of these properties! Importantly, all such (expected) L-function had to do with objects of arithmetic nature.

This is where Hecke comes in, around 1936. He discovered a completely different way of producing such functions from holomorphic modular forms which were eigenfunctions of what we now call Hecke operators. Namely, given $f(q) = \sum a_n q^n$, you can write down the L-function $L(f, s) = \sum a_n n^{-s}$. The property of having an Euler product goes back to Ramanujan, but the proof that this has an analytic continuation and a functional equation was due to Hecke. He knew that Riemann's proof of the functional equation of the zeta function used the θ-function, and this was probably important motivation for him. Hecke's crucial observation was that if you view the holomorphic modular form f as an automorphic form $f : [\mathrm{PGL}_2] \to \mathbb{C}$, then one has

$$\int_{[\mathbb{G}_m]} f(h)|h|^s dh = L(f,s).$$

Don't worry about the character $|h|^s$ in this expression; up to this issue, it is precisely the pairing of f with the period function $\Theta_{\mathrm{PGL}_2,\mathrm{PGL}_2/\mathbb{G}_m}$. This is amazing, because it is an analytic, as opposed to arithmetic, source for such L-functions.

After Hecke, people started discovering several other ways to produce L-functions through analytic methods. The next example, in the 1940s, was due to Rankin and Selberg. They produced an L-function from a pair f, g of Hecke eigenforms. If we write $f(q) = \sum a_n q^n$ and $g(q) = \sum b_n q^n$, then the Rankin-Selberg L-function was $\sum_{n\geq 1} \frac{a_n b_n}{n^s}$. That it has an Euler product is not so hard, but that it has an analytic continuation or functional equation is completely nonobvious. The way that they proved these things was, in our language, given by pairing $f \boxtimes g$ (viewed as an automorphic form for $G = \mathrm{GL}_2 \times \mathrm{GL}_2$) with $\Theta_{G,X}$ where $X = \mathrm{GL}_2 \times \mathbb{A}^2$. Here, the two factors of GL_2 act on GL_2 on the left and right, and only the second copy of GL_2 acts on $\mathbb{A}^2$. Rankin's motivation was not to study prime numbers; instead, he was interested in bounding the absolute value $|a_n|$. This proof was very important, because it provided Deligne with inspiration for proving some key inequalities in the proof of the Weil conjectures.

After this, there was a while before there were more examples. But Godement and Jacquet, around 1970, produced an L-function starting from a Hecke eigenform f on GL_n. Again, they proved that it has an analytic continuation and functional equation by proving that this L-function is given by pairing f with $\Theta_{G,X}$, where $G = \mathrm{GL}_n \times \mathrm{GL}_n$ and $X = \mathrm{Mat}_n$. In all of these papers, what you will find are integrals and manipulations with them; it was much later that the conceptualization as the study of a certain space came in. After this, the theory of automorphic forms took up, and many, many generalizations of the above were found.

Around the 1960s, Langlands proposed that *all* L-functions (including all arithmetic ones) come from automorphic forms. This was important because there were a lot of examples and tools to work with L-functions coming from automorphic forms, but almost none for arithmetic ones. This became a very strong motivation to develop the theory of L-functions for automorphic forms.

Let us emphasize that all these examples due to Hecke, Rankin-Selberg, Godement-Jacquet, etc., are of the following form. One considers the pairing $\langle f, \Theta_{G,X}\rangle$, which turns out to be either 0, *or* gives an L-function related to f. It turns out that the conditions under which the pairing $\langle f, \Theta_{G,x}\rangle$ was nonzero are nicely packaged by the dual Hamiltonian $\check{G}$-space $\check{M}$. In more detail, whether this pairing vanishes has to do with the dual group $\check{G}_X$; and the L-function that one obtains when the pairing is non-vanishing has to to with V_X, the representation that appeared in the Plancherel measure. Also, $\Theta_{G,X}$ only depends on the cotangent bundle $M = T^*X$. So a large number of examples are packaged by a pair M, $\check{M}$. However, not all known examples fall within the framework of hyperspherical duality. This is the nicest class of such examples, but it would be desirable to have some sort of extension of the framework.

13. Day 5 Lecture 1

13.1. **Recap.** Recall that we want to identify two theories, aka. automorphic theory (of G) and spectral theory (of $\check{G}$) in various settings (global arithmetic; global geometric, local arithmetic; local geometric). In particular, boundary theories, e.g. L-functions and periods should match. For example, Fargues-Scholze formulation of local Langlands correspondence over arithmetic local field (e.g. $\mathbb{Q}_p$) shares simlarity with geometric Langlands over a curve.

In the local arithmetic setting, recall that $C_c^\infty(X_G)^{G_\mathbb{O}}$ corresponds to the dual space $\check{M}$ with $\check{G}$ action. Categorifying it, we have local geometric conjecture, relating

$$\mathrm{Shv}(X_F/G_\mathbb{O}) \cong \mathrm{QCoh}^{/\!/}(\check{M}/\check{G}).$$

Example 13.1. For the group case where $G \times G$ acts on $X = G$, on the dual side we have $\check{G} \times \check{G}$ acting on $\check{X} = \check{G}$ and hence $\check{M} = T^*\check{G} \cong \check{G} \times \check{\mathfrak{g}}^*$. The local geometric conjecture recovers the derived geometric Satake equivalence: the A-side is $\mathrm{Shv}(G_O\backslash G_F/G_O)$, and the B-side is $\mathrm{QCoh}^{/\!/}(\check{\mathfrak{g}}^*/\check{G})$. The B-side is identified with the assignment of TQFT to the ravioli via Koszul duality. More precisely we have

$$\mathcal{B}_{\check{G}}(D \coprod_{D^\times} D) = \mathrm{QCoh}^!(\check{\mathfrak{g}}[-1]/\check{G}) \cong \mathrm{QCoh}^{/\!/}(\check{\mathfrak{g}}^*/\check{G}).$$

Here the first identity is what we get from the field description of the TQFT and the last equivalence is Koszul duality.

Today our goal is to discuss global geometric conjecture and then understand the arrows connecting global geometric conjecture to local geometric and global arithmetic conjecture.

13.2. **Setup.** Fix Σ a smooth projective algebraic curve over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$.

There is a group G acting on X and we attach a period sheaf $\mathcal{P}_X \in \mathrm{Shv}(\mathrm{Bun}_G\Sigma)$. Dually $\check{G}$ acts on $\check{X}$ and we attach to it an L-sheaf $\mathcal{L}_{\check{X}} \in \mathrm{QCoh}^!(\mathrm{Loc}_{\check{G}}\Sigma)$. Unramfied geometric Langlands match the two categories and the relative Langlands further match the objects, aka. the period sheaf should match the L-sheaf.

Remark 13.2. Relative Langlands duality provides us with much more matching of objects than that was used in the formulation and proof of GLC.

13.3. Bun_G^X **and period sheaf.** Consider $\mathrm{Bun}_G^X = \mathrm{Bun}_G^X(\Sigma) = \mathrm{Maps}(\Sigma, X/G)$, which is a stack classifying G-bundles on Σ with sections of associated X-bundles. This stack is equipped with a natural map

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G. \tag{13.3}$$

Definition 13.4. We define **period sheaf** by the formula

$$\mathcal{P}_X := \pi_! \underline{k}_{\mathrm{Bun}_G^X}.$$

Example 13.5. Take $G = \mathbb{G}_m$ and then

$$\mathrm{Bun}_G = \mathrm{Pic}_\Sigma \cong \mathbb{Z} \times \mathrm{Jac}\Sigma \times B\mathbb{G}_m.$$

Example 13.6. Consider $\mathrm{Bun}_{\mathrm{SL}_2}(\mathbb{P}^1)$. It is not compact. It is connected, and its points correspond to $\mathcal{O} \oplus \mathcal{O}$, $\mathcal{O}(1) \oplus \mathcal{O}(-1), \ldots$. These get equipped with the topology such that $\mathcal{O}(a) \oplus \mathcal{O}(-a)$ is in the closure of $\mathcal{O}(b) \oplus \mathcal{O}(-b)$ if $0 \leq b \leq a$.

13.4. **Local Systems.** Dual to Bun_G, we consider

$$\mathrm{Loc}_{\check{G}}^{\check{X}}\Sigma := \mathrm{Maps}_{k\text{l.c.}}(\Sigma, \check{X}/\check{G}), \tag{13.7}$$

which is equipped with a map

$$\check{\pi} : \mathrm{Loc}_{\check{G}}^{\check{X}}\Sigma \to \mathrm{Loc}_{\check{G}}\Sigma := \mathrm{Maps}_{k\text{l.c.}}(\Sigma, \mathrm{pt}/\check{G}), \tag{13.8}$$

where the subscript k-l.c. means locally constant maps.

Example 13.9. For $G = \mathbb{G}_m$, then

$$\mathrm{Loc}_{\check{G}} = B\mathbb{G}_m \times (\mathbb{G}_m)^{2g} \times (\text{derived stuff}), \tag{13.10}$$

where g is the genus of Σ. More canonically, the factor $(\mathbb{G}_m)^{2g}$ can be thought as $H^1(\Sigma, \mathbb{C}^\times)$.

13.5. ***L*-sheaf and its fibers.** We define L-sheaf corresponding to a $\check{G}$-space $\check{X}$ and study its fibers. Recall that we had a map $\check{\pi} : \mathrm{Loc}_{\check{G}}^{\check{X}} \to \mathrm{Loc}_{\check{G}}$.

Definition 13.11. We define ***L*-sheaf** by the formula $\mathcal{L}_{\check{X}} := \check{\pi}_* \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}} \in \mathrm{QC}^!(\mathrm{Loc}_{\check{G}})$.

Remark 13.12. We have an isomorphism between sheaves $\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}$ and $\mathcal{O}_{\mathrm{Loc}_{\check{G}}^{\check{X}}}$.

Remark 13.13. Why do we use the dualizing sheaf instead of the structure sheaf? It is roughly because the category IndCoh privileges dualizing sheaf whereas the category QCoh privileges structure sheaf.

Remark 13.14. We have to normalize the L-sheaf, which on the level of functions amounts to normalizing L-functions to make sense of inner product.

Given a local system $E \in \mathrm{Loc}_{\check{G}}$, the fiber of $\check{\pi}$ at E has the following equivalent descriptions:

(1) locally constant sections of the associated $\check{X}$ bundle of E,
(2) E-twisted locally constant maps $\Sigma \to \check{X}$,
(3) $\pi_1(\Sigma)$-fixed points on $\check{X}$.

In the last description, we are using the action of $\pi_1(\Sigma)$ on $\check{X}$ given by the pullback of the action of $\check{G}$ on $\check{X}$ via $E : \pi_1(\Sigma) \to \check{G}$. Indeed, all of the descriptions above should be interpreted in the derived sense. Denote the fiber of $\check{\pi}$ at E by $\check{\pi}^{-1}E$.

Remark 13.15. We might ask: given a group H acting on a space Y, what does it mean by its derived fixed locus? Classically, the fixed locus can be interpreted as the space of maps from pt/H to Y. We make sense of the derived fixed locus by considering the derived mapping space from pt/H to Y.

From its definition, given a local system E, the fiber of $\mathcal{L}_{\check{X}}$ at E is $R\Gamma(\check{\pi}^{-1}E, \omega)$. Hence $\mathcal{L}_{\check{X}}$ is coherent on the locus where $\mathrm{R}\,\Gamma(\check{\pi}^{-1}E, \omega)$ is finite-dimensional, or equivalently if the number of (classical) fixed points is finite. This can also be phrased as (classical) fixed points being isolated.

Example 13.16. Suppose $\check{X}$ is a linear representation of $\check{G}$. Having isolated fixed points is equivalent to not having a trivial subrepresentation of $\pi_1\Sigma$ acting on $\check{X}$. This condition determines the locus on $\mathrm{Loc}_{\check{G}}$ where the L-sheaf $\mathcal{L}_{\check{X}}$ is coherent.

13.6. **Global geometric conjecture.** Now we state the BZSV global geometric conjecture. We first state a naive version.

Conjecture 13.17 (Naive global geometric conjecture)**.** *The period sheaf $\mathcal{P}_X$ and L-sheaf $\mathcal{L}_{\check{X}}$ match under the geometric Langlands correspondence:*

$$\begin{aligned} \mathbb{L}_G : \mathrm{Shv}(\mathrm{Bun}_G) &\to \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}), \\ \mathcal{P}_X &\mapsto \mathcal{L}_{\check{X}}. \end{aligned}$$

Recall that we had three choices of sheaf theory on the A-side, namely de Rham, Betti, and étale. Except for the de Rham setting, this conjecture should be modified: for the statement of GLC, sheaves on the A-side should satisfy a certain condition $\mathcal{N}$. This is natural from the TQFT point of view. From the local constancy axiom of TQFT, the action of Hecke category (which is $\mathcal{A}_G(D_x \coprod_{D_x^\times} D_x)$) on the category of sheaves on Bun_G (which is $\mathcal{A}(\Sigma)$) should be locally constant on $x \in \Sigma(F)$. Not every sheaves over Bun_G satisfy this requirement, hence we should replace the A-side and consider a subcategory $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ to make this local constancy work. Hence the modified statement of GLC for Betti or étale setting would be an equivalence

$$\mathbb{L}_G : \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G) \to \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}).$$

Remark 13.18. This condition $\mathcal{N}$ is not needed in de Rham setting, because every D-modules are locally constant in a weaker sense.

The following two examples of $G = \mathbb{G}_m$ illustrate the necessity of the condition $\mathcal{N}$ from two perspectives. For both of the examples, we work on the Betti setting.

Example 13.19 (TQFT)**.** Let $G = \mathbb{G}_m$. We have $\mathrm{Bun}_G = \mathrm{Pic}$. Given $n \in \mathbb{Z}$ and a point $x \in \Sigma(F)$ we have an automorphism

$$m_x^n : \mathrm{Pic} \to \mathrm{Pic}, \quad \mathcal{L} \mapsto \mathcal{L}(nx).$$

Consider the weight n representation of $\operatorname{Rep}\mathbb{G}_m^\vee$ viewed as an object in the spherical Hecke category at the point $x \in \Sigma(F)$. The action of such an object on Shv(Pic) is identified with

$$(m_x^n)_* : \mathrm{Shv}(\mathrm{Pic}) \to \mathrm{Shv}(\mathrm{Pic}).$$

The space Pic has $\mathbb{Z}$-many components labeled by degrees, and from the local constancy condition on the Hecke action, we see that $\mathcal{F} \in \mathcal{A}_G(\Sigma)$ should be locally constant on each component.

Example 13.20 (Direct computation)**.** Recall that $\mathrm{Pic} = \mathrm{Jac} \times (\cdots)$ and $\mathrm{Loc}_{\mathbb{G}_m} = H^1(\Sigma, \mathbb{C}^\times) \times (\cdots)$, where $(\cdots)$ are some extra factors. For simplicity, let us discard the extra factors from both sides. We have

$$\mathrm{QCoh}(H^1(\Sigma, \mathbb{C}^\times)) = \mathcal{O}(H^1(\Sigma, \mathbb{C}^\times)) - \mathrm{mod}\,.$$

Fourier transformation gives us an isomorphism $\mathcal{O}(H^1(\Sigma, \mathbb{C}^\times)) \cong \mathbb{C}[H_1(\Sigma, \mathbb{Z})]$ where the right-hand side should be understood as a group algebra. Hence we arrive at

$$\mathrm{QCoh}(\mathrm{Loc}_{\mathbb{G}_m}) \cong H_1(\Sigma, \mathbb{Z}) - \mathrm{mod}\,.$$

On the other hand, $H_1(\Sigma, \mathbb{Z}) = \pi_1(\mathrm{Jac})$, and therefore we confirm the equivalence

$$\mathrm{Loc}(\mathrm{Jac}) \cong \mathrm{QCoh}(\mathrm{Loc}_{\mathbb{G}_m}).$$

From this, we conclude that GLC forces sheaves on the A-side to be locally constant. This agrees with our conclusion from the previous example.

So far, we have modified the GLC. To state the BZSV conjecture we still have to make another modification, because period sheaves do not necessarily belong to the category $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$. Still, we have a *Beilinson projector* $(-)^{\mathrm{spec}} : \mathrm{Shv}(\mathrm{Bun}_G) \twoheadrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ so we might expect the following.

Conjecture 13.21 (Global geometric conjecture)**.** *The projection of period sheaf $\mathcal{P}_X$ and L-sheaf $\mathcal{L}_{\check{X}}$ match under the geometric Langlands correspondence:*

$$\begin{aligned} \mathbb{L}_G : \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G) &\to \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}), \\ (\mathcal{P}_X)^{\mathrm{spec}} &\mapsto \mathcal{L}_{\check{X}}. \end{aligned}$$

Remark 13.22. We ignored some technical issues such as normalization and duality involution. A more precise form of this conjecture can be found in Conjecture 12.1.1 of BZSV.

13.7. **More on the Beilinson projector.** To explain the necessity of the Beilinson projector for period sheaves, let us look at some examples of period sheaves and L-sheaves.

Example 13.23. If we let $X = \mathrm{pt}$, the case which we refer as *Neumann boundary condition*, the map $\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G$ is an identity, therefore $\mathcal{P}_{\mathrm{pt}} = \underline{k}_{\mathrm{Bun}_G}$. If we let $\check{X} = \mathrm{pt}$, then in the same way $\mathcal{L}_{\mathrm{pt}} = \omega_{\mathrm{Loc}_{\check{G}}}$. Note that there is a trivialization of $\omega_{\mathrm{Loc}_{\check{G}}}$ coming from a symplectic structure of $\mathrm{Loc}_{\check{G}}$. This symplectic structure is canonically defined using a Killing form of $\mathfrak{g}^\vee$.

Example 13.24. If we take the G-space $X = G$, the case which we refer to as *Dirichlet boundary condition*, the map $\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G$ is an inclusion $\mathrm{pt} \hookrightarrow \mathrm{Bun}_G$ of the trivial bundle. Hence $\mathcal{P}_G$ is the skyscraper sheaf at the trivial bundle. In the same way, $\mathcal{L}_{\check{G}}$ is a skyscraper sheaf at the trivial local system.

Remark 13.25. If G is abelian, those two examples are dual to each other through the Fourier transform.

In the Example 13.24, the period sheaf $\mathcal{P}_G$ is a skyscraper, hence it is far from being locally constant. This tells us that period sheaves do not necessarily live in the category $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$, explicating the necessity of the Beilinson projector.

Remark 13.26. The L-sheaves do not generally land in the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$. Note that we are using a 'bigger' version of GLC allowing every ind-coherent sheaves on the B-side.

The condition $\mathcal{N}$ and the Beilinson projector may seem like an unpleasant feature of global geometric conjecture. However, measurements in $\mathrm{Shv}_{\mathcal{N}}$ that we are interested in can be lifted to measurements in Shv via adjunction. There is an adjunction between the projection $(-)^{\mathrm{spec}} : \mathrm{Shv}(\mathrm{Bun}_G) \twoheadrightarrow \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G)$ and the inclusion $\mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G) \hookrightarrow \mathrm{Shv}(\mathrm{Bun}_G)$. In the Betti setting, the projection is left-adjoint to the inclusion, while in the étale setting the projection is right-adjoint to the inclusion. Using this adjunction, in the Betti setting we have

$$\mathrm{Hom}_{\mathrm{Shv}_{\mathcal{N}}}((\mathcal{P}_X)^{\mathrm{spec}}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Shv}}(\mathcal{P}_X, \mathcal{F}) \tag{13.27}$$

for a period sheaf $\mathcal{P}_X$ and a Hecke eigensheaf $\mathcal{F}$. An analogous statement can be made for étale setting if we consider Hom spaces the other way around. Note that in number theory, the pairing between period and automorphic function is the measurement we are mainly interested in on the A-side. Hence number theory *does not* sense the difference between the categories Shv and $\mathrm{Shv}_{\mathcal{N}}$.

We end the discussion of the Beilinson projector giving a simple example.

Example 13.28. Recall from Example 13.24 that $\mathcal{P}_G = i_! k$ where $i : \mathrm{pt} \hookrightarrow \mathrm{Bun}_G$ is an inclusion of the trivial bundle. As pointed out earlier, this doesn't satisfy the condition $\mathcal{N}$ and therefore needs to be projected to the category $\mathrm{Shv}_{\mathcal{N}}$. The idea is to replace pt with a homotopy equivalent space C (i.e., a contractible space) so that we can take !-pushforward along a map $j : C \to \mathrm{Bun}_G$ instead. If we choose the replacement C so that the map $C \to \mathrm{Bun}_G$ is a fibration, then the sheaf $j_! \underline{k}_C$ is locally constant, hence lies in $\mathrm{Shv}_{\mathcal{N}}$. We let $(\mathcal{P}_G)^{\mathrm{spec}} = j_! \underline{k}_C$.

13.8. **Sanity check for the Iwasawa-Tate case.** One of the most important examples of relative Langlands duality is the Iwasawa-Tate case, where $G = \mathbb{G}_m$ acts on $X = \mathbb{A}^1$ and $\check{G} = \mathbb{G}_m$ acts on $\check{X} = \mathbb{A}^1$. The proof of Conjecture 13.21 for this case was given in Theorem 6.1.2 of Feng-Wang "Geometric Langlands duality for periods". We wouldn't go through its proof, but instead do a simple sanity check by comparing Hom spaces of both sides. In this subsection, we work on the Betti setting exclusively. A more detailed account of this sanity check can be found in Sect. 12.2.3 of BZSV.

Let E be a *non-trivial* $\mathbb{G}_m$-local system on Σ. It determines a skyscraper sheaf $\underline{\mathbb{C}}_E \in \mathrm{QC}^!(\mathrm{Pic})$. Under the GLC, it corresponds to a rank one $\mathbb{C}$-linear local system $\mathcal{F}_E$ on Pic. The local system $\mathcal{F}_E$ has a property that for each $r \geq 0$, its pullback under the map

$$\phi_r : \Sigma^r \to \mathrm{Pic}, \quad (P_1, \cdots, P_r) \mapsto \mathcal{O}(P_1 + \cdots + P_r)$$

is $\mathcal{E}^{\boxtimes r}$. Here $\mathcal{E}$ is the rank one $\mathbb{C}$-linear local system associated to E. Hence Conjecture 13.21 predicts

$$\mathrm{Hom}_{\mathrm{Shv}_{\mathcal{N}}}((\mathcal{P}_{\mathbb{A}^1})^{\mathrm{spec}}, \mathcal{F}_E) \cong \mathrm{Hom}_{\mathrm{QC}^!(\mathrm{Loc}_{\mathbb{G}_m})}(\mathcal{L}_{\mathbb{A}^1}, \underline{\mathbb{C}}_E).$$

From the adjunction (13.27), it is equivalent to

$$\mathrm{Hom}_{\mathrm{Shv}}(\mathcal{P}_{\mathbb{A}^1}, \mathcal{F}_E) \cong \mathrm{Hom}_{\mathrm{QC}^!(\mathrm{Pic})}(\mathcal{L}_{\mathbb{A}^1}, \underline{\mathbb{C}}_E). \tag{13.29}$$

Now we give a proof of this.

Using adjunction, the right-hand side of (13.29) is the dual of fiber of $\mathcal{L}_{\mathbb{A}^1}$ at E. As explained in Subsection 13.5, the fiber of $\mathcal{L}_{\mathbb{A}^1}$ at E is $R\Gamma(\check{\pi}^{-1}E, \omega)$ where $\check{\pi}^{-1}E$ is the derived fixed locus of the action $\pi_1(\Sigma)$ on $\mathbb{A}^1$. From Remark 13.12, the derived section of dualizing sheaf is identified with the function algebra of $\check{\pi}^{-1}E$. Combining these,

$$\mathrm{Hom}_{\mathrm{QC}^!(\mathrm{Pic})}(\mathcal{L}_{\mathbb{A}^1}, \underline{\mathbb{C}}_E) \cong \mathcal{O}(\check{\pi}^{-1}E)^*. \tag{13.30}$$

Since we assumed that E is non-trivial, the derived stack $\check{\pi}^{-1}E$ has a single classical point: a zero section. Hence its function algebra is determined by tangent complex. The tangent complex is $H^1(\Sigma, \mathcal{E})[-1]$ and hence

$$\mathcal{O}(\check{\pi}^{-1}E) = \mathrm{Sym}(H^1(\Sigma, \mathcal{E})^*[1]). \tag{13.31}$$

Combining (13.30) and (13.31) we get

$$\mathrm{Hom}_{\mathrm{QC}^!(\mathrm{Loc}_{\mathbb{G}_m})}(\mathcal{L}_{\mathbb{A}^1}, \underline{\mathbb{C}}_E) \cong \mathrm{Sym}(H^1(\Sigma, \mathcal{E})[-1]). \tag{13.32}$$

Now we compute the left-hand side of (13.29). The map $\check{\pi} : \mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1} \to \mathrm{Pic}$ admits a section $i : \mathrm{Pic} \to \mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1}$ which amounts to choosing a zero section of the associated $\mathbb{A}^1$-bundle. The section i is a closed embedding, and we may consider the complementary open embedding

$j : \mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1,0} \to \mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1}$. Note that the open substack $\mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1,0}$ classifies a pair $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle over Σ and s is its non-trivial section. Taking zeros of the section with multiplicity, this stack is identified with the stack $\mathrm{Sym}\,\Sigma$ of effective divisors. Hence we have a distinguished triangle

$$\underline{\mathbb{C}}_{\mathrm{Pic}} \to \mathcal{P}_{\mathbb{A}^1} \to j_! \underline{\mathbb{C}}_{\mathrm{Sym}\,\Sigma}. \tag{13.33}$$

We claim that $\mathrm{Hom}_{\mathrm{Shv}}(\underline{\mathbb{C}}_{\mathrm{Pic}}, \mathcal{F}_E) = 0$. Recall that Pic consists of $\mathbb{Z}$-many copies of $\mathrm{Jac} \times B\mathbb{G}_m$ (Example 13.5) and that Hecke operators permute those copies (Example 13.19). Since the local system $\mathcal{F}_E$ is a non-trivial Hecke eigensheaf, the previous facts tell us that $\mathcal{F}_E$ is non-trivial on each copy of $\mathrm{Jac} \times B\mathbb{G}_m$. Hence for the proof of the claim, it suffices to show that $\mathrm{Hom}_{\mathrm{Jac} \times B\mathbb{G}_m}(\underline{\mathbb{C}}_{\mathrm{Jac} \times B\mathbb{G}_m}, \mathcal{F}) = 0$ for any non-trivial rank one local system $\mathcal{F}$ on $\mathrm{Jac} \times B\mathbb{G}_m$. This is equivalent to a vanishing statement of group cohomologies of the monodromy representation of $\mathcal{F}$. Since Jac is a torus, the claim follows from the fact that the group cohomology groups $H^*(\Lambda, k_\chi)$ vanish for an free abelian group Λ of finite rank and its non-trivial character k_χ.

Combining the claim with the distinguished triangle (13.33) we get

$$\mathrm{Hom}_{\mathrm{Shv}}(\mathcal{P}_{\mathbb{A}^1}, \mathcal{F}_E) \cong \mathrm{Hom}_{\mathrm{Shv}}(j_! \underline{\mathbb{C}}_{\mathrm{Sym}\,\Sigma}, \mathcal{F}_E).$$

Using adjunction and the fact that j is an open embedding, this can be rewritten as

$$\mathrm{Hom}_{\mathrm{Shv}}(\mathcal{P}_{\mathbb{A}^1}, \mathcal{F}_E) \cong R\Gamma(\mathrm{Sym}\,\Sigma, j^* \mathcal{F}_E). \tag{13.34}$$

Recall that $\mathcal{F}_E$ pulls back to $\mathcal{E}^{\boxtimes r}$ via $\phi_r : \Sigma^r \to \mathrm{Pic}$, hence

$$R\Gamma(\Sigma^r, \phi_r^* \mathcal{F}_E) = R\Gamma(\Sigma, \mathcal{E})^{\otimes r} = (H^1(\Sigma, \mathcal{E})[-1])^{\otimes r}.$$

From this, we may expect

$$R\Gamma(\mathrm{Sym}\,\Sigma, j^* \mathcal{F}_E) \cong \mathrm{Sym}(H^1(\Sigma, \mathcal{E})[-1]).$$

This combined with (13.34) gives us

$$\mathrm{Hom}_{\mathrm{Shv}}(\mathcal{P}_{\mathbb{A}^1}, \mathcal{F}_E) \cong \mathrm{Sym}(H^1(\Sigma, \mathcal{E})[-1]). \tag{13.35}$$

We pass the sanity check (13.29) by comparing (13.32) with (13.35).

14. Day 5 Lecture 2

DISCLAIMER: I have tried to reproduce the lecture faithfully, but apologize for any inaccuracies or omissions I may have introduced. In particular, I have found it difficult to reproduce words that were verbally spoken but not written down.

Notational remarks: In the interest of compiling the combined lecture notes from different note-takers, I will be using $(-)^{\mathrm{sh}}$ for the shearing notation ("slanted box") in BZSV. Note that the meaning of this notation was not precisely defined in the lectures, although it was said roughly that it turns non-zero cohomological degrees into $\mathbb{G}_m$-grading.

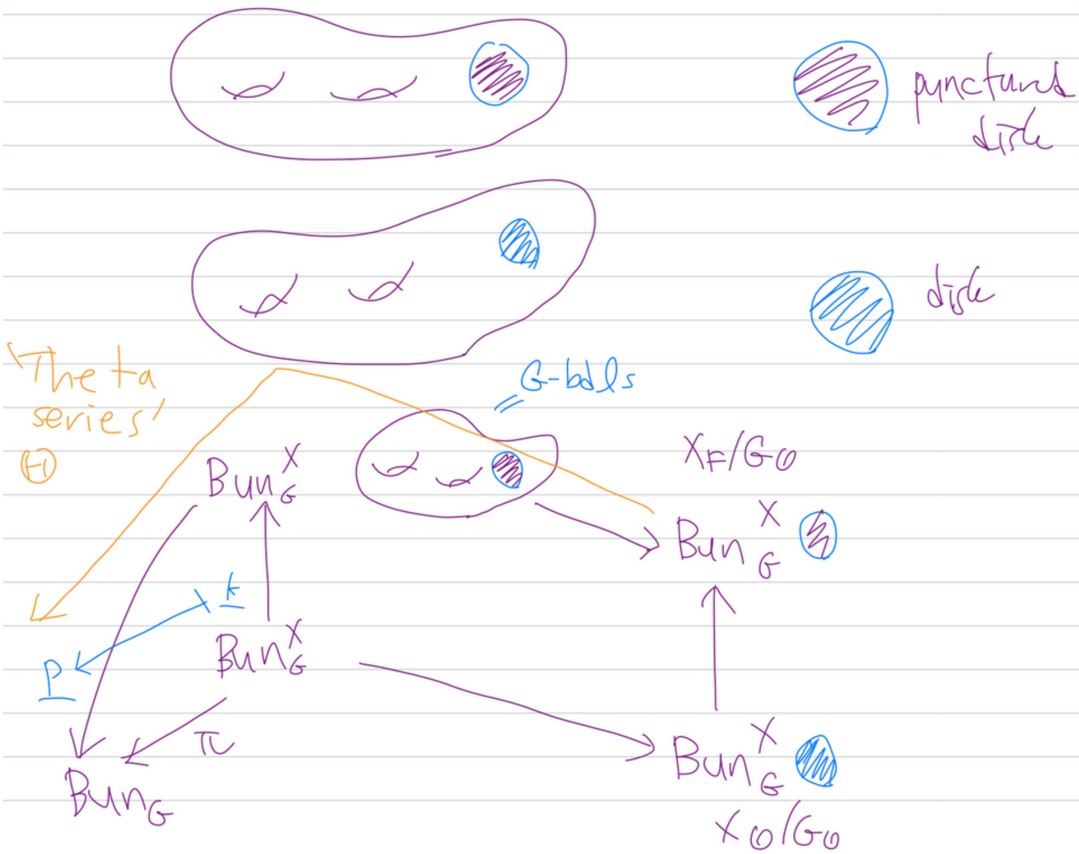
Conjecture 14.1. *Under the Geometric Langlands Correspondence,*[10]

$$\begin{aligned} &\mathrm{Shv}(\mathrm{Bun}_G) \overset{()^{\mathrm{spec}}}{\twoheadrightarrow} \mathrm{Shv}_{\mathcal{N}}(\mathrm{Bun}_G) \simeq \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}) \\ &\mathcal{P} \mapsto \mathcal{L}. \end{aligned}$$

For questions regarding Hom's,

$$\mathrm{Hom}(\mathcal{P}^{\mathrm{spec}}, \text{Hecke eigensheaves}) = \mathrm{Hom}(\mathcal{P}, \text{“automorphic forms”})$$

is insensitive to the spectral projection $()^{\mathrm{spec}}$ above.



$$\begin{aligned} &\otimes_{s\in S}\mathrm{Shv}(X_F/G_{\mathcal{O}}) \xrightarrow{\Theta} \mathrm{Shv}(\mathrm{Bun}_G(\Sigma)) \\ &\delta_0 := \underline{k}_{X_{\mathcal{O}}/G_{\mathcal{O}}} \text{ (basic object)} \mapsto \mathcal{P}_X = \Theta(\delta_0) \text{ (period sheaf)} \end{aligned}$$

The map above is compatible with the action of $\mathcal{H}$ on both sides.

At the level of functions:

$$C_c^\infty(X_F/G_{\mathcal{O}}) \xrightarrow{\Theta} \mathrm{Fun}(\{G-\text{bundles w/ trivialization at } s\}) \ni \mathcal{P} = \Theta(\delta_0)$$

[10]The (rough) conjecture below was not fully written during the lecture, and I have added $\mathcal{L}$ based on my understanding.

There is a fully ramified version of the above:

$$\begin{gathered}\mathrm{Shv}(X_F) \to \mathrm{Shv}(\mathrm{Bun}_{G,s}) \\ \circlearrowright\ G_F \circlearrowright\end{gathered}$$

$$\begin{gathered}C_c^\infty(X_F) \xrightarrow{\Theta} \mathrm{Fun}(\mathrm{Bun}_G(\mathbb{F}_q)) \\ \circlearrowright\ G_F \circlearrowright\end{gathered}$$

In the polarized case $\check{M} = T^*\check{X}$, we define the corresponding L-sheaves as follows:

$$\begin{aligned}&\mathrm{QC}^{\mathrm{sh}}(\check{M}/\check{G}) \to \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}\Sigma) \\ &\mathcal{O} \mapsto \mathcal{L}_{\check{X}}\end{aligned}$$

Note that the LHS is equivalent to $\mathrm{QC}^!(\mathrm{Loc}_{\check{G}}^{\check{X}}(\text{punctured disc}))$, and the object corresponding to $\mathcal{O}$ above is the dualizing sheaf ω.

Example 14.2.

$$\begin{aligned}&G \circlearrowright G/H = X, X/G = \cdot/H \\ &\mathrm{Bun}_H = \mathrm{Bun}_G^X \xrightarrow{\pi} \mathrm{Bun}_G \\ &\mathcal{P}_{G/H} = \pi_! \underline{k}_{\mathrm{Bun}_H} \\ &\mathrm{Hom}(\mathcal{P}_{G/H}, \mathcal{F}) = \Gamma(\mathrm{Bun}_H, \pi^! \mathcal{F}) \\ &\mathcal{P}_H(f) = \int_{[H]} f|_{[H]}, \ \ f \text{ aut. form on } [G]\end{aligned}$$

Example 14.3.

$$G \circlearrowright G/N = X \circlearrowright T, X/(G \times T) = \cdot/B$$

$$\begin{tikzcd} & \mathrm{Bun}_{G\times T}^X = \mathrm{Bun}_B \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_T \end{tikzcd}$$

From this correspondence, we get the Eisenstein series functor

$$\begin{tikzcd} \mathrm{Eis} : \mathrm{Shv}(\mathrm{Bun}_T) \arrow[r] & \mathrm{Shv}(\mathrm{Bun}_G) \arrow[l, bend right, dotted, "\text{Constant Term}"'] \end{tikzcd}$$

This is an integral transform represented by $\mathcal{P}_{G\circlearrowright G/N \circlearrowright T}$. The duality here will be

$$G \circlearrowright G/N \circlearrowright T \Longleftrightarrow \check{G} \circlearrowright \check{G}/\check{N} \circlearrowright \check{T}$$

$$\mathrm{Shv}(\mathrm{Bun}_G) \xLeftarrow{\mathrm{Bun}_B}\!\!\Longrightarrow \mathrm{Shv}(\mathrm{Bun}_T)$$

$$\Longleftrightarrow$$

$$\mathrm{QC}^!(\mathrm{Loc}_{\check{G}}) \xLeftarrow{\mathrm{Loc}_{\check{B}}}\!\!\Longrightarrow \mathrm{QC}^!(\mathrm{Loc}_{\check{T}})$$

Remark 14.4. Period sheaves are *very* far from Hecke eigensheaves. On the spectral side, this is roughly the difference between a skyscraper sheaf vs. an $\mathcal{L}$-sheaf.

Example 14.5 (Group case)**.** Spherical variety

$$G \times G \circlearrowright G \Longleftrightarrow \check{G} \times \check{G} \circlearrowright \check{G} \text{ (modulo a Chevalley twist)}$$

This corresponds to the boundary conditions:

$$G \overset{\mathrm{id}}{|} G \longrightarrow \check{G} \overset{\mathrm{id}}{|} \check{G}$$

$$\begin{array}{c} \mathrm{Bun}_{G\times G}^{X=G} \simeq \mathrm{Bun}_G \\ {\scriptstyle\Delta}\big\downarrow \\ \mathrm{Bun}_G \times \mathrm{Bun}_G = \mathrm{Bun}_{G\times G} \end{array}$$

$$\mathcal{P}_{\text{group}} = \Delta_! \underline{k}, \mathcal{L}_{\text{group}} = \Delta_! \omega.$$

This encodes Gaitsgory's Miraculous Duality, which says that $\mathrm{Shv}(\mathrm{Bun}_G)$ has a version of Verdier duality matching the Serre duality on the spectral side.

Example 14.6.

$$\mathbb{G}_m \circlearrowright \bullet \text{ (Neumann)} \Longleftrightarrow \mathbb{G}_m \circlearrowright \mathbb{G}_m \text{ (Dirichlet)}$$

The $\mathbb{G}_m$ action on $\mathbb{G}_m$ on the RHS is by (left) multiplication[11], and this is a non-spherical example.

Example 14.7. $M = T^*G//_f N \circlearrowright G$ $(f \in \mathfrak{n}^*)$, $G \circlearrowright \bullet$

$$\mathcal{P}_\bullet = \underline{k}_{\mathrm{Bun}_G} \longleftrightarrow \mathcal{L}_{\mathrm{Whit}} \text{ (spectral Whittacker sheaf)}$$

Whittacker normalization:

$$\begin{aligned} &\mathcal{P}_{\mathrm{Whit}} \longleftrightarrow \mathcal{L}_\bullet = \mathcal{O}_{\mathrm{Loc}_{\check{G}}} = \omega_{\mathrm{Loc}_{\check{G}}} \\ &\mathrm{Hom}(\mathcal{L}_\bullet, -) = \mathrm{R\Gamma}(\mathrm{Loc}_{\check{G}}, -) \\ &\mathrm{Hom}(\mathcal{P}_{\mathrm{Whit}}, -) = \mathrm{R\Gamma}(\mathrm{Bun}_N, \text{against a character}). \end{aligned}$$

[11]added: of course, left or right does not matter since $\mathbb{G}_m$ is abelian

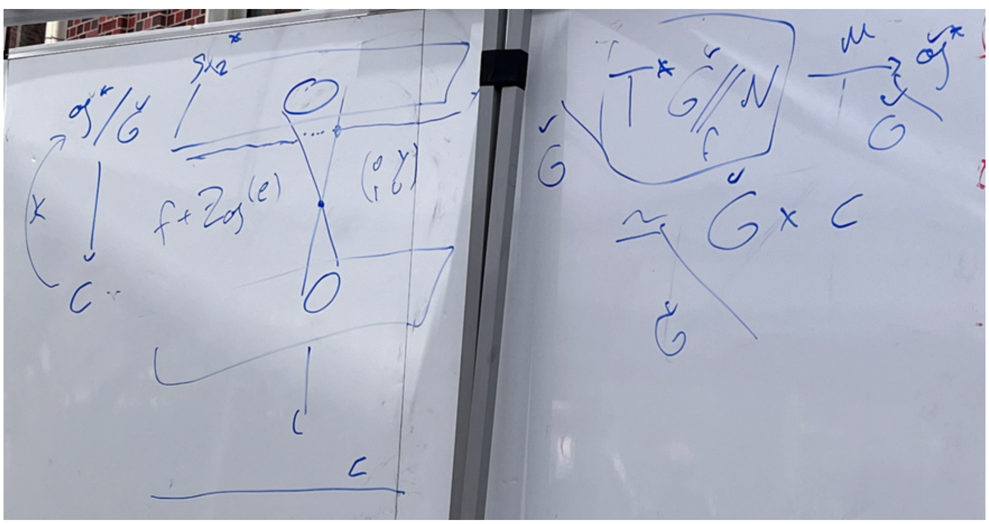
Example 14.8. [12]

$$\Sigma = \text{“raviolo”} = D \sqcup_{D^*} D$$
$$\mathcal{P}_\bullet = \underline{k}_{\mathrm{Gr}_G}$$
$$\mathrm{Hom}(\mathcal{P}_\bullet, -) = H^*(\underline{\mathrm{Gr}}_G, -) = H^*_{G(\mathcal{O})}(\mathrm{Gr}_G, -).$$
$$\mathfrak{c} = \mathfrak{g}//G = \mathfrak{h}//W \simeq \check{\mathfrak{h}}^*//W$$

$$\begin{tikzcd}
\mathrm{Shv}(\mathrm{Bun}_G(\text{raviolo})) \arrow[rr, "\simeq"] \arrow[d] & & \mathrm{QC}^{\mathrm{sh}}(\check{\mathfrak{g}}^*/\check{G})) \arrow[dll, "\mathcal{K}^*"] \\
H^*_G(\bullet) - \mathrm{mod} = \mathbb{C}[G] - \mathrm{mod}
\end{tikzcd}$$

(Back to Whittacker case) $\check{G}\backslash T^*\check{G}//_f N \xrightarrow{\mu} \check{G}\backslash\check{\mathfrak{g}}^* \simeq \check{G}\backslash\check{G} \times \mathfrak{c}$.

Picture for $\mathfrak{sl}_2^*$: The Kostant section κ is a section of the characteristic polynomial map $\check{\mathfrak{g}}^*/\check{G} \to \mathfrak{c}$ given by $f + Z_{\mathfrak{g}}(e)$.

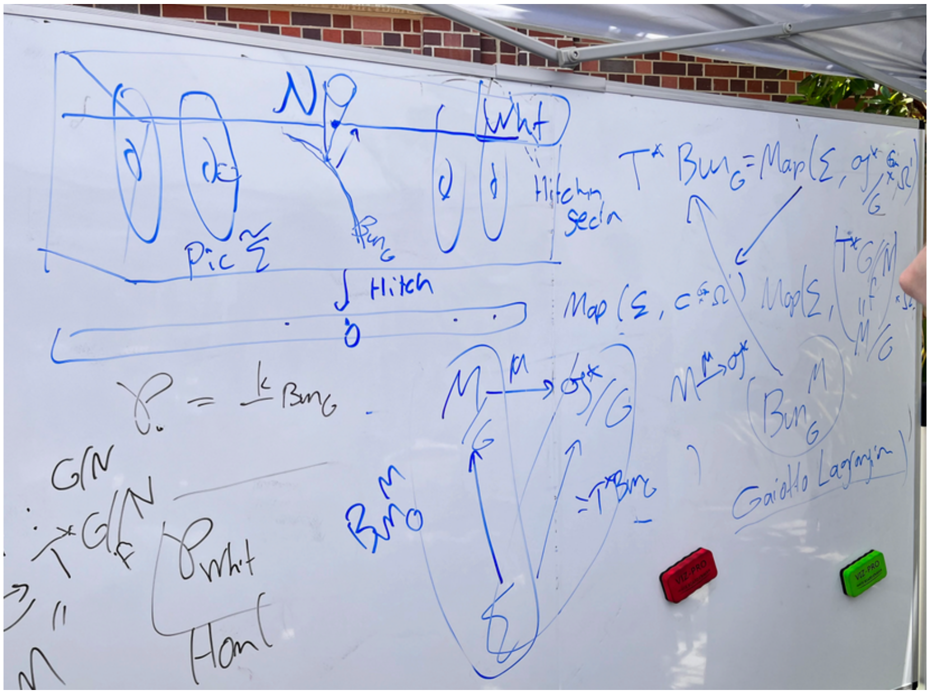


The Hitchin section is the global analogue of the Kostant slice. It intersects the global nilpotent cone transversely. The Whittacker sheaf is the constant sheaf on the Hitchin section.

$$T^*\mathrm{Bun}_G = \mathrm{Map}(\Sigma, \mathfrak{g}^*/G \overset{\mathbb{G}_m}{\times} \Omega^1) \to \mathrm{Map}(\Sigma, \mathfrak{c} \overset{\mathbb{G}_m}{\times} \Omega^1)$$

The Gaiotto Lagrangian is the (image of the) natural morphism $\mathrm{Bun}_G^M = \mathrm{Map}(\Sigma, M/G) \to \mathrm{Map}(\Sigma, \mathfrak{g}^*/G)$, induced by the moment map (modulo G) $\mu : M/G \to \mathfrak{g}^*/G$.

[12] added: Below, note that $\underline{\mathrm{Gr}}_G$ with the underline denotes not the affine Grassmanian Gr_G, but the Hecke stack $G_{\mathcal{O}}\backslash G_F/G_{\mathcal{O}}$.



15. Day 5 Lecture 3

Ben-Zvi has explained $\theta_{G,X} \leftrightarrow \theta_{\check{G},\check{X}}$ when we put a curve over $\overline{\mathbb{F}_q}$.

In this lecture, I want to go back to the case of a curve Σ over $\mathbb{F}$, and explain the formulas of the type

$$\langle P_X, f\rangle = \begin{cases} 0, & \text{some cases} \\ \text{a certain L-functions, else} \end{cases} \tag{15.1}$$

can be formulated in a way that aligns with Ben-Zvi's explanation and with the TQFT picture.

In this lecture we will suppose that $M = T^*X$, $\check{M} = T^*\check{X}$ are a pair of hyperspherical spaces, both polarized, although everything today can be done with M and $\check{M}$ alone. The following discussion is very schematic, and we will skip all manner of assumptions and details, see BZSV paper for a careful version with proper caveats and everything pinned down.

There were many formulas of the form (15.1). In BZSV, we rewrite these formulas in the form

$$\langle P_X, f\rangle = \sum_{x\in \check{X}^{\mathrm{fix}}} L(T_x).$$

The right hand side is symmetrically indexed by a space, zero comes from the absence of fixed points on these space, the L-function comes from the tangent of the same space. In the TQFT language, it says that $P_X \in \mathcal{A}_G(\Sigma)$ matches with the right-hand side, considered as a function on the set Loc_Σ.

On the left hand side, f is a Hecke eigenform, it corresponds to a single point on the dual side - a homomorphism $\rho : \pi_1(\Sigma) \to \check{G}$. Therefore $\pi_1(\Sigma)$ acts on $\check{X}$ and we can take the fixed points. On the right hand side, we sum over fixed points x for this ρ, T_x is the tangent space, which carries a π_1-representation. And **L-functions are geometrized by exterior algebra on cohomology**, that is $L(T_x)$ equals the trace of Frobenius on

$$\sum_j (-1)^j \wedge^j H^1(\Sigma_{\overline{\mathbb{F}_q}}, T_x).$$

Remark 15.2. This is not the ususal definition of L-function, but it is related to it by the Grothendieck-Lefschetz trace formula (which expresses the L-function as a characteristic polynomial of Frobenius) and for g an $n \times n$ matrix

$$\det(1-g) = \sum_{j=0}^{n} (-1)^j \mathrm{trace}(g| \wedge^i \mathbb{C}^n)$$

where the usual definition is the left-hand side.

We have the local geometric conjecture: there is an equivalence of categories

$$\mathrm{Sh}(X_F/G_{\mathcal{O}}) \longrightarrow \text{Quasi-coherent sheaves on } \check{M}/\check{G}$$

where the trace of Frobenius on derived Hom's categorified the inner product comes from the local plancherel formula

$$\langle T_V\delta_0, T_W\delta_0\rangle = \int_{\check{G}_{compact}} \chi_V \overline{\chi}_W (\text{q-character of } \check{M})\; d\mu_{\mathrm{Haar}}. \tag{15.3}$$

Under the geometric Langlands correspondence, we have

$$\mathrm{Shv}(Bun_G) \cong \mathrm{QC}^!(\mathrm{Loc}_{\check{G}})$$
$$\underline{P}_X \leftrightarrow \underline{L}_{\check{X}}$$

Now we are going to geometrize the pairings by Hom as in local Plancherel formula 15.3

$$\langle P_X, f\rangle = \sum_{x \in \check{X}^{\mathrm{fix}}} L(T_x)$$

on the geometric side, f corresponds to a skyscraper, geometric conjecture implies that

$$\mathrm{Hom}(\underline{P}_X, \underline{f}) = \mathrm{Hom}(\underline{L}, \underline{\delta}_f).$$

Left hand side looks good, but why does fiber of the L-sheaf geometrize $\sum_x L(T_x)$?

The fibers of the L-sheaf above ρ concerns functions/forms on *derived fixed points* of $\rho : \pi_1 \to \check{G}$ acting on $\check{X}$.

Example 15.4. When Γ acts on a vector space W

- usual fixed points are $W^\Gamma = H^0(\Gamma, W)$ with ring of functions a symmetric algebra.
- derived fixed points record $H^i(\Gamma, W)$ for all $i \geq 1$, e.g. $H^1(\Gamma, W)$ contributes an exterior algebra to the function ring.

From the discussion in the previous example, we get

$$\text{fiber of L-sheaf} = \bigoplus_{x \in \check{X}^{\text{fix}}} \wedge^* H^1(T_x)$$

and $\wedge^* H^1$ geometrizes the L-function.

Remark 15.5. Under certain assumptions, the fibers of the L-sheaves are calculated in section 11.8 of BZSV paper.

Next, we talk about the connection between geometry and arithmetic: In principle geometrical statements for a curve $\Sigma_{\overline{\mathbb{F}_q}}$ imply numerical statements for a curve $\Sigma_{\mathbb{F}_q}$. In practice the two settings have different strengths and inform each other. Geometry buys something very important, the symmetry of the two sides.